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Twisted partition functions for ADE boundary conformal field theories and Ocneanu algebras of quantum symmetries

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We dedicate this article to the memory of our friend Prof. Juan A. Mignaco deceased, 6 June 2001.

Abstract

For every ADE Dynkin diagram, we give a realization, in terms of usual fusion algebras (graph algebras), of the algebra of quantum symmetries described by the associated Ocneanu graph. We give explicitly, in each case, the list of the corresponding twisted partition functions.

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1. Introduction

For each ADE Dynkin diagram G, we consider the corresponding Ocneanu graph $\mathrm{Oc}(G)$, as given by Ocneanu [16], and build explicitly an algebra (the algebra of quantum symmetries of the given Dynkin diagram) whose multiplication table is encoded by this Ocneanu graph. Using this algebra structure, we obtain explicitly, and easily, the expression of all the twisted partition functions that one may associate with the given Dynkin diagram (one for each vertex of its corresponding Ocneanu graph).

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Our first purpose is not to deduce the graph $\mathrm{Oc}(G)$ from G, since that was already done by Ocneanu [16] (actually the details have never been made available in printed form), but to give a simple presentation of the corresponding algebra of quantum symmetries. In each case, this algebra will be given in terms of a quotient of the tensor square of the graph algebra (fusion algebra) associated with some particular ADE Dynkin diagram. These algebras $\mathcal{H}_{\mathrm{Oc}(G)}$ are commutative in almost all cases (for D_{2n} they involve 2×2 matrices).

Our other purpose is to use this structure to obtain explicitly the corresponding "toric matrices" (terminology taken from [6]) and the corresponding "twisted partition functions" (terminology taken from [21,22]).

The torus structure of all ADE models has been worked out by Ocneanu himself several years ago (unpublished). Explicit expressions for the eight toric matrices of dimension 6×6 of the D_4 model can be found in [21] (where they are interpreted physically in terms of the 3-state Potts model) and for the twelve toric matrices of dimension 11×11 of the E_6 model in [6] (where one can also find a general method of calculation for these quantities). One of our purposes is to give explicit results, in particular for all exceptional cases, by following the method explained in this last reference [6] and summarized in Section 2. Starting from conformal field theory (CFT), another general method for obtaining the structure of these twisted partition functions has been described in the subsequent article [22] which contains closed formulae; we do not use this formalism. Actually, the constructions performed in the sequel avoid, deliberately, the use of CFT concepts.

Again, we insist upon the fact that we take for granted the data given by the Ocneanu graphs themselves; otherwise, we should either have to diagonalize the convolution product of the quantum Racah–Wigner bi-algebra associated with the given ADE diagram, or to solve the problem of finding what are the irreducible elements for the set of "connections" that one can define on a pair of graphs (system of generalized Boltzman weights, see also [24]). This was done by Ocneanu himself. The present paper can be read independently of [6] since all the necessary information is gathered in Section 2.

As already stressed, our own presentation, which follows [6], uses neither the language nor the techniques of CFT, but the results themselves can be interpreted in terms of CFT. For instance the toric matrices lead to quantities that can be interpreted in terms of partition functions for boundary conformal field theories in presence of defect lines. The reader interested in those CFT aspects should look at the article [22] which contains many results of independent interest and is probably the most complete published work on this subject, in relation with conformal field theories.

For every ADE example, the particular toric matrix associated with the "unit vertex" of the corresponding Ocneanu graph is the usual modular invariant for the associated ADE model (in the classification of [4]), i.e., the corresponding sesquilinear form gives the usual modular invariant partition function. The other partition functions (the non-trivially "twisted" ones), those associated with the other points of the Ocneanu graph are not modular invariant.

It is unfortunately almost impossible to provide a unified (or uniform) treatment for all ADE diagrams; indeed, all of them are "special", in one way or another. The A_n are a bit too "simple" (many interesting constructions just coincide in that case), the D_{2n} are the only ones to give rise to a non-abelian algebra of quantum symmetries, the E_7 does not define a positive integral graph algebra and the D_{2n+1} do not define any integral graph algebra at all; "only" E_6 and E_8 lead, somehow, to a similar treatment.

The structure of the present paper is as follows: after a first section devoted to a general overview of the theory, we examine separately all types of ADE Dynkin diagrams. In each case, i.e., in each section, after having presented the graph algebra associated with the chosen diagram (when it exists), we describe explicitly the structure of an associative algebra that we can associate with its corresponding Ocneanu graph, express it in terms of (usual) graph algebras and deduce, from this algebra structure, the corresponding toric matrices. In order not to clutter the paper with sparse matrices of big size, we list only the sesquilinear forms—i.e., the twisted partition functions—associated with these toric matrices. For pedagogical reasons we prefer to perform this analysis in the following order: A_n, E_6, E_8, D_{2n}, D_{2n+1}, E_7.

2. Summary of the algebraic constructions

2.1. Foreword

To every pair of ADE Dynkin diagrams with the same Coxeter number, one may associate (Ocneanu) an algebra of quantum symmetries. Its elements (also called "connections" on the given pair of graphs) can be added and multiplied in a way analogous to what is done for representation of groups; in particular, this algebra has a unit, and one may consider a set of "irreducible" quantum symmetries, that, by definition, build up a basis of linear generators for this algebra (an analogue of the notion of irreducible representations). Using multiplication, we may also single out, in each case, two (algebraic) generators, usually called "chiral left" and "chiral right" generators, playing the role of fundamental representations for groups: all other irreducible elements can be obtained as linear combinations of products of these two generators. The Ocneanu graph precisely encodes this algebraic structure: its number of vertices is equal to the number of irreducible elements and edges encode multiplication by the two generators.

When the two chosen Dynkin diagrams coincide, we can find another interpretation for the Ocneanu graph (and algebra) of quantum symmetries. This is actually the case of interest, for us, in the present paper. Here is a sketch of the theory. One first considers elementary paths (i.e., genuine paths) on the chosen Dynkin diagram G; one then build the Hilbert space $\mathrm{Path}(G)$ of all paths, by taking linear combinations of elementary paths and declaring that elementary paths are orthogonal. This vector space provides a path model for the Jones algebra associated with G. The next step is to consider the vector subspace $\mathrm{EssPath}(G)$ of essential paths: by definition, they span the intersection of kernels of all Jones projectors; in the classical situation, the essential paths starting from the origin would correspond to projectors on the symmetric representations of $SU(2)$ (or of finite subgroups of $SU(2)$). We refer to the paper [7] (to be contrasted with [6]) where a geometrical study of the classical binary polyhedral groups [14] (symmetries of Platonic bodies) is performed, using McKay correspondence [15], by studying paths and essential paths on the affine Dynkin diagrams of type $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$. Essential paths start somewhere (a), end somewhere (b) and have a certain length (n). The finite-dimensional vector space $\mathrm{EssPath}(G)$ is therefore graded by the length n of the paths: $\mathrm{EssPath}(G) = \oplus_n \mathrm{EssPath}^n(G)$. Notice that essential paths are usually linear combinations of elementary paths. We may then build the graded algebra $\mathcal{A} \doteq \oplus_n \mathrm{End}(\mathrm{EssPath}^n(G))$, where each summand is the space of endomorphisms of $\mathrm{EssPath}^n(G)$; they can be explicitly written as square matrices. The algebra $\mathcal{A}$ is not only an algebra (for the obvious composition $\circ$ of endomorphisms) but also a *bi*-algebra: using concatenation of elementary paths together with the existence of a scalar product on $\mathrm{Path}(G)$, one can define a convolution product $*$ on $\mathcal{A}$. Details concerning this construction, also due to Ocneanu, and about its interpretation in the case of affine ADE diagrams (i.e., in the case of $SU(2)$ itself and the usual polyhedral groups) will be found in [8]. $\mathcal{A}$ is therefore a kind of finite-dimensional and quantum analogue of the Racah–Wigner bi-algebra. Being semi-simple for both algebra structures $\circ$ and $*$, we may decompose $\mathcal{A}$ as a sum of square matrices (blocks) in two different ways. For the first structure ($\circ$), which is obvious from its very definition, the corresponding projectors are labelled by n (the length of essential paths). For the second structure ($*$), the blocks are labelled by an index, that we shall call x; the Ocneanu algebra is then precisely the algebra spanned by those x, i.e., by the corresponding projectors: this is an analogue of the table of multiplication of characters (convolution product) for a finite group. The underlying vector space of $\mathcal{A}$ possesses two adapted basis, one is expressed in terms of the “double triangles of Ocneanu” (that we prefer to draw as a “fermionic” diffusion graph with a connecting vertical “photon” line labelled by n), the other in terms of diffusion graphs with horizontal “very thick lines” labelled by x, the vertices of the Ocneanu graph. The change of basis between the two adapted basis can be thought of as a duality relation; it is a kind of generalized Fourier transform involving quantum Racah symbols at a particular root of unity depending on the chosen Dynkin diagram. It will also be conceptually important to consider the length n as labelling a particular vertex of an A_N graph (the first vertex to the left being labelled 0).

Several constructions used in our paper can certainly be understood in terms of planar algebras [12] (see also [11]), nets of subfactors [2,3], or in terms of braided categories [13], but we shall not discuss this here. We do not plan, in the present paper, to give any interpretation of these constructions in terms of standard Hopf algebra constructions: this has not been worked out, yet.

2.2. Structure of the following sections

2.2.1. The diagram (ADE) and its adjacency matrix

We give the diagram G itself, choose a particular labelling for vertices and give the adjacency matrix $\mathcal{G}$ in a specified basis. We consider vertices σ_v of G as would be irreducible representations for a quantum analogue of a group algebra $\mathcal{H}_G$ that we do not need to define. We also write down the norm β of G (the biggest eigenvalue of $\mathcal{G}$) and the Perron–Frobenius eigenvector D (i.e., the normalized eigenvector corresponding to β, with its smallest component, associated with the vertex σ_0, normalized to the integer 1). The components of D give (by definition) the quantum dimensions of the irreducible representations σ_v. In all cases, β is equal to the q-number [1] $[2]_q = q + 1/q = 2\cos(\pi/\kappa)$. This value κ is, by definition, the Coxeter number of the graph. In all cases, the q-dimension of σ_0 (the marked vertex) is $[1]_q = 1$. More information can be gathered, for instance, from the book [10].

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[1] We define $[n]_q = (q^n - q^{-n})/(q - q^{-1})$, where $q = \exp(\mathrm{i}\pi/\kappa)$.

We should remember the values of Coxeter numbers for the various ADE Dynkin diagrams:

	A_n	D_{n+2}	E_6	E_7	E_8
Coxeter number	$n+1$	$2(n+1)$	12	18	30

2.2.2. The graph algebra of the Dynkin diagram and the quantum table of characters

The next step is to associate with the diagram G, when possible, a commutative algebra playing the role of an algebra of characters. This algebra is linearly generated, as a vector space, by the vertices σ_v of G. As an associative algebra, it admits a unit σ_0 and one generator σ_1, with quantum dimension $[2]_q$. The relations of this associative algebra are defined by the graph G itself, considered as encoding multiplication by the generator σ_1: the irreducible representations appearing in the decomposition of $\sigma_1\sigma$, with σ, a vertex of G, are the neighbours of σ on the diagram G. We impose, furthermore, that the structure constants of this algebra should be positive integers, as it is the case for irreducible representations of groups or, more generally, of Hopf algebras. It is (almost well) known, since [19] that, for ADE diagrams, the solution to the above problem does not exist for E_7 and D_{odd}. For all other ADE diagrams, there exists a unique solution. This algebra is called the graph algebra associated with G, or the fusion algebra associated with G and sometimes [25], the dual Pasquier algebra of G. Such a commutative algebra is also a "positive integral hypergroup", or simply an hypergroup, when no confusion arises (see [1] and references therein). We shall denote this algebra by the same symbol as the graph itself, and hope that no confusion with the simple Lie group bearing the same name will arise. Practically, we have to build a multiplication table, the first two rows and columns being already known (multiplication by the unit σ_0 and by the generator σ_1). The table is built in a very straightforward way, by imposing associativity. For instance, in the case of the graph A_n, $n > 4$, let us calculate,

$$\begin{aligned}\sigma_2\sigma_2 &= (\sigma_1\sigma_1 - \sigma_0)\sigma_2 = \sigma_1\sigma_1\sigma_2 - \sigma_0\sigma_2 = \sigma_1(\sigma_1 + \sigma_3) - \sigma_2\\ &= \sigma_0 + \sigma_2 + \sigma_2 + \sigma_4 - \sigma_2 = \sigma_0 + \sigma_2 + \sigma_4.\end{aligned}$$

In every case (except E_7 and D_{odd}) we shall give the multiplication table of the graph algebra. When writing down this table, and in order to save space, we shall drop the symbols σ and refer to the different vertices only by their subscript.

The graph matrix algebra of the ADE diagram G, with r vertices, is a matrix algebra linearly generated by r matrices of size $r \times r$ providing a faithful realization of the graph algebra spanned by the σ_a's. Its construction is straightforward: to σ_0 one associates the unit matrix (call it G_0) and to the generator σ_1, we associate a matrix G_1 equal to the adjacency matrix $\mathcal{G}$ of the diagram; to the other vertices σ_a, expressed in terms of σ_0 and σ_1 we associate the corresponding matrices G_a given in terms of G_0 and G_1. Since these last two matrices are already explicitly known, in order to save space, we shall just give the polynomial expressions giving the G_j in terms of these two.

In the particular case of A_N graphs, the fusion matrices G_i will be also called N_i.

The r matrices G_a commute with one another (they are all polynomials in one and the same G_1) and can be simultaneously diagonalized thanks to a matrix S_G. If the σ_a's were irreducible representations of a finite group, this matrix S_G would be the table of characters for this finite group, i.e., the result of the pairing between conjugacy classes and irreducible characters (notice that we do not need to build explicitly the conjugacy classes!). In the present situation, S_G is the quantum analogue of a table of characters. In the case of A_N graphs, the same matrix, simply denoted by S, represents one of the generators of the modular group $SL(2, \mathbb{Z})$ (Verlinde–Hurwitz representation).

2.2.3. Essential matrices and paths

The general definition of essential paths on a graph was defined by Ocneanu [16], but we do not need this precise definition here because we just need to count these particular paths. We shall nevertheless recall this definition in the appendix. It is enough to know that the general notion of essential paths generalizes the notion of symmetric (or q-*symmetric*) representations (at least for those paths starting from the origin). Some general comments and particular cases (diagrams E_6 and $E_6^{(1)}$) can be read in [6,7]. The number $E_a[p, b]$ of essential paths of length p starting at some vertex a and ending on the vertex b is given by the component of the row vector $E_a(p)$ defined as follows:

- $E_a(0)$ is the (line) vector characterizing the chosen initial vertex;
- $E_a(1) \doteq E_a(0) \cdot \mathcal{G}$;
- $E_a(p) \doteq E_a(p-1) \cdot \mathcal{G} - E_a(p-2)$.

The expression of $E_a(0)$ depends on the chosen ordering of vertices; it is convenient anyway to set $E_0(0) = (1, 0, 0, \dots)$ for the unit σ_0 of G, and $E_1(0) = (0, 1, 0, \dots)$ for the generator σ_1. For a graph with r vertices, starting from $E_a(0)$, we *would* obtain in this way r rectangular matrices E_a with infinitely many rows (labelled by $p = 0, 1, 2, \dots$) and r columns (labelled by b).

The reader can check that, for Dynkin ADE diagrams, the numbers $E_a(p)$ are positive integers *provided* $0 \le p \le \kappa - 2$ (κ being the Coxeter number of the graph), but this ceases to be true as soon as $p > \kappa - 2$. For instance, in the case of the E_6 graph, where $\kappa = 12$, we get $E_0(11) = (0, 0, 0, 0, 0, 0)$, $E_0(12) = (0, 0, 0, 0, -1, 0)$. This reflects the fact [16] that essential paths on these graphs, with a length bigger than $\kappa - 2$, do not exist. We call “essential matrices” the r rectangular $(\kappa - 1) \times r$ matrices obtained by keeping only the first $\kappa - 1$ rows of the $E_a(\cdot)'s$. For every ADE diagram, these finite-dimensional rectangular matrices will still be denoted[2] by E_a. The components of the rectangular matrix E_a are denoted by $E_a[p, b]$. Matrix elements of these matrices can be displayed as vertices with three edges labelled by a, b, p (or, dually, as triangles). Warning: the smallest value for p, the length of essential paths, is 0, and not 1.

In order to save space, we shall not give explicitly all these matrices E_a, although they are absolutely crucial for obtaining the next results; however their calculation, using the above recurrence formulae, is totally straightforward, once the matrix $G_1 \equiv \mathcal{G}$ is known. The pattern of non-zero entries of an essential matrix E_a, associated with some graph G, gives a figure expressing “visually” the structure of the space of essential paths starting

[2] Not to be confused with the symbol used for the exceptional Dynkin diagrams themselves!

from a. These essential matrices were introduced in [6] as a convenient tool, but the geometrical patterns themselves were first obtained by Ocneanu (the essential paths starting from all possible vertices are displayed, for all ADE graphs, in the appendix of [16]). In the following sections, we shall only display the essential matrix E_0 that encodes essential paths starting from the origin. Let us finally mention that a list of the rectangular matrices E_0's (not the E_a's), interpreted in the context of RSOS models, can be found in [23].

When the diagram is not an ADE but an *affine* ADE, the essential matrices are no longer of finite size: they have infinitely many rows; they can be interpreted in terms of the classical induction–restriction theory for representations of $SU(2)$ and its finite subgroups (binary polyhedral groups) (see the lecture notes [7] for a study of the corresponding classical geometries, along the above lines).

For Dynkin diagrams of type A_N, we have the relation $N_i = N_{i-1} \cdot \mathcal{G} - N_{i-2}$, and from the above definition of essential matrices, we see that there is no difference, in this case, between the fusion graph matrices $G_i = N_i$ and the essential matrices E_i.

Before ending this section, we should point out the fact that since rows of the essential matrices associated with a particular Dynkin diagram G have labels p running from 0 to $\kappa - 1$, they are therefore also indexed by the vertices τ_p of the Dynkin diagram $A_{\kappa-1}$. In this way, we can interpret these essential matrices as a kind of quantum analogue of the theory of induction/restriction: irreducible representations of $A_{\kappa-1}$ can be "reduced" to irreducible representations of G (essential matrices can be read "horizontally" in this way) and irreducible representations of G can "induce" irreducible representations of $A_{\kappa-1}$ (essential matrices can be read "vertically" in this way). Rather than displaying the essential matrices, or the corresponding spaces of paths, we shall only give, for each vertex of the graph G, the list of induced representations of $A_{\kappa-1}$. This information can be deduced immediately from the essential matrix E_0. In other words, we consider, for each vertex σ_v of G an associated quantum vector bundle and decompose the space of its sections into irreducible representations of $A_{\kappa-1}$.

2.2.4. *Dimensions of blocks for the Racah–Wigner–Ocneanu bi-algebras*

The Racah–Wigner–Ocneanu bi-algebra $\mathcal{A}$ is a direct sum of blocks in two different ways (see Section 2.1). Its dimension is obtained either by summing the squares d_n^2, where d_n is the number of essential paths of length n or by summing the squares d_x^2, where the d_x are the sizes of the Ocneanu blocks. The integers d_n are obtained by summing all matrix elements of the row $n + 1$ over all essential matrices E_a (all vertices a of a given diagram). This first calculation is relatively easy.

The integers d_x giving the number of "vertices" labelled by (a, b, x) can be obtained from the multiplication table of $\mathcal{H}_{\mathrm{Oc}(G)}$. If the label x of an Ocneanu block is of the type $a\dot{\otimes}b$, or a linear combination of such elements (the notation $\dot{\otimes}$ is introduced later in the text), and when $\mathcal{H}_{\mathrm{Oc}(G)}$ is commutative and contains two (left and right) subalgebras isomorphic with the graph algebra of the Dynkin diagram G, the integers $d_x = d_{a\dot{\otimes}b}$ can be obtained simply by summing all matrix elements $(\Sigma_x)_d^c$ of the matrices $\Sigma_x \dot{=} G_a G_b$, where G_a and G_b are fusion matrices of the Dynkin diagram G. This holds in particular for A_N and for the exceptional cases E_6 and E_8. The other cases—in particular the case of E_7—are slightly more involved. We refer to the corresponding sections.

The knowledge of integers d_n, d_x was implicit in the work of Ocneanu, already presented to several audiences years ago (for instance [18]). The values of d_n and d_x were first published, for the E_6 case, in [6] (the treatment of the E_8 case being the same). General results, for all cases, were published in [22]. We take advantage of the explicit realization that we find for the bialgebra $\mathcal{H}_{\mathrm{Oc}(G)}$ to recover easily all the results, including the more difficult E_7 case (see the corresponding section).

We give the integers d_n, d_x and the sums $\sum d_n$, $\sum d_x$ and $\sum d_n^2 = \sum d_x^2$. The equality of squares is a direct consequence of the bi-algebra structure. In most cases (not D_{even}) one finds also that $\sum d_n = \sum d_x$; this can be understood as coming from a change of basis in the vector space $\mathrm{EssPath}(G)$. The equality of sums can actually be also achieved for D_{even} by performing the summation only on particular classes of elements (see the discussion made in [22].)

2.2.5. The Ocneanu graph corresponding to a Dynkin diagram and its algebra

The Ocneanu graph $\mathrm{Oc}(G)$ associated with a Dynkin diagram G was already discussed in Section 2.1. As already stated, we take it directly from reference [16].

One of our purposes is to give an explicit presentation for the corresponding algebras, that will be called $\mathcal{H}_{\mathrm{Oc}(G)}$. In most cases it will be obtained from the tensor square of some graph algebra, by taking the tensor product over a particular subalgebra (not over the complex numbers). The multiplication is the natural one, namely: $(a_1 \otimes b_1) \times (a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$, and we shall identify $au \otimes b$ and $a \otimes ub$, whenever u belongs to the particular subalgebra over which the tensor product is taken (we use the notation $\dot{\otimes}$). In other words, we take the quotient of the tensor square of the appropriate graph algebra by the two-sided ideal generated by elements $0 \otimes u - u \otimes 0$, where 0 is the unit of the graph algebra of G. In the cases of D_{odd} and E_7, the above construction has to be “twisted”: some elements $au \otimes b$ have to be identified with $a \otimes \rho(u)b$, but ρ is not the identity map.

In most cases, the graph algebra to be used in the above construction is the graph algebra of G itself. In the case of the diagram E_7, however, one has to use the graph algebra of D_{10}. For the diagram D_{2n+1} one has to use the graph algebra of A_{4n-1}. For the diagram D_{2n}, elements of $\mathcal{H}_{\mathrm{Oc}(D_{2n})}$ also involve 2×2 matrices.

In general, the elements u that are used to define the appropriate two-sided ideal belong to a subalgebra U that admits a complementary subspace P which is invariant by left and right U-multiplications (a general feature since the algebra U is semi-simple). This property implies that elements of $\mathcal{H}_{\mathrm{Oc}(G)}$ can be decomposed into linear combinations of only four types of elements belonging to $0\dot{\otimes}U$, $0\dot{\otimes}P$, $P\dot{\otimes}0$ and $P\dot{\otimes}P$.

Following Ocneanu terminology, we call “chiral left subalgebra” or “chiral right subalgebra” the subalgebras spanned by left or right generators ($\sigma_1 \otimes \sigma_0$ or $\sigma_0 \otimes \sigma_1$) and “ambichiral” the intersection of the chiral parts. Left and right subalgebra are respectively described on Ocneanu graphs by fat continuous lines, and fat dashed lines. The thin lines (continuous or dashed) represent right or left cosets.

Warning. A given algebra $\mathcal{H}_{\mathrm{Oc}(G)}$ is, in a sense, already defined by its graph $\mathrm{Oc}(G)$ since the later describes multiplication by the two chiral generators. What we do in this paper is to propose, for all Dynkin diagram G, an explicit realization of these algebras $\mathcal{H}_{\mathrm{Oc}(G)}$, in terms of usual graph algebras. In turn, this realization allows us in a simple way to determine all the toric matrices associated with a given diagram (see below). We stress the fact that the quantum graphs $\mathrm{Oc}(G)$ are taken from [16], however the proposed realizations for the algebras $\mathcal{H}_{\mathrm{Oc}(G)}$ are ours.

The number of vertices of $\mathrm{Oc}(G)$ depends very much of the choice of G itself (for instance $\mathrm{Oc}(E_6)$ contains 12 points, $\mathrm{Oc}(E_7)$ contains 17 points, $\mathrm{Oc}(E_8)$ contains 32 points).

2.2.6. Modular invariant partition functions and twisted partition functions

To every vertex x of the Ocneanu graph $\mathrm{Oc}(G)$ of the Dynkin diagram G, one associates a particular "toric matrix" W_x. These matrices are related to the study of paths on the Ocneanu graphs: the matrix element $(W_x)_{i,j}$ of W_x gives the number of independent paths leaving the vertex x of $\mathrm{Oc}(G)$ and reaching the origin $0\dot{\otimes}0$ of $\mathrm{Oc}(G)$ after having performed i essential steps (respectively j essential steps) on the left (respectively right) chiral subgraphs. These matrices have other uses and interpretations (in particular in terms of the cell calculus or in terms of the "chiral modular splitting" [17] but this will not be discussed here.

As written in Section 1, these toric matrices were defined and obtained by Ocneanu (unpublished but advertised in several conferences since 1995, for instance [18]). Reference [22] gives closed formulae for the determination of these objects, in the language of CFT. One of our purposes, in the present paper, is to find them by another method, which consists in a straightforward generalization of the technique introduced in [6]. This method uses explicitly our realization of the algebras $\mathcal{H}_{\mathrm{Oc}(G)}$ in terms of graph algebras.

Our first step is to compute the appropriate essential matrices (those of the graph associated with the graph algebra involved in the previous step); generally, i.e., not for E_7 or D_{odd}, these are the r essential matrices of the graph G itself. As discussed previously, they are rectangular matrices E_a of size $(\kappa-1)\times r$. We then construct "reduced essential matrices" E_a^{red} by keeping only those columns associated with the subalgebra over which the tensor product is taken (i.e., we replace the other entries by 0). These are again rectangular matrices of size $(\kappa-1)\times r$.

We then define matrices $W[a,b]$ associated with elements $x = a\otimes b$ of the algebra $\mathcal{H}_{\mathrm{Oc}(G)}$ as square matrices of size $(\kappa-1)\times(\kappa-1)$ by setting

$$W[a,b]\dot{=}E_a\cdot\tilde{E}_b^{\mathrm{red}} \equiv (E_a)\cdot\mathrm{transpose}(E_b^{\mathrm{red}}) = (E_a)^{\mathrm{red}}\cdot\mathrm{transpose}(E_b^{\mathrm{red}}).$$

The points x of $\mathrm{Oc}(G)$ are, in general, linear combinations of elements of the type $a\dot{\otimes}b$. The toric matrices W_x associated with points $x=\sum a\dot{\otimes}b$ of $\mathrm{Oc}(G)$ are square matrices of size $(\kappa-1)\times(\kappa-1)$. They are obtained by setting

$$W_x = \sum W[a,b].$$

In the case of E_7 and D_{odd} the above construction should be slightly twisted (see the relevant sections for details).

There are several ways to display the results: one possibility is to give the collection of all toric matrices $W_x = W[a,b]$ with matrix elements $W[a,b](i,j)$, another one is to fix i and j (with $1\le i,j\le\kappa-1$) and display the Ocneanu graph itself labelled by the entries $W[a,b](i,j)$. For physical reasons (at least for traditional reasons) we prefer to display the corresponding (twisted) partition functions: setting $\chi \doteq \{\chi_0, \chi_1, \chi_2, \dots, \chi_{\kappa-2}\}$, we associate with $W_x = W[a,b]$ a partition function

$$Z_x \equiv Z[a,b] \doteq \bar{\chi} W[a,b] \chi .$$

To ease the reading of the paper we put all these partition functions in tables to be found at the end of the article. The matrix elements of all $W[a,b]$ are always positive integers. However, in order to display the results in tables, we had sometimes to group together several terms and introduce minus signs that will disappear if the sesquilinear forms are expanded.

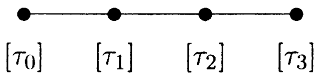
These quantities can be interpreted in terms of twisted partition functions for ADE boundary conformal field theories (see [3] [20,22]).

The partition function $Z[0,0]$ associated with the origin $\sigma_0 \otimes \sigma_0$ is the usual modular invariant partition function of Itzykson, Capelli and Zuber. The others are not modular invariant. We should remember, at that point, that the representation of the modular group provided by the usual S and T matrices, in the representation of Verlinde–Hurwitz, is usually not effective: for instance in the case of E_6, where $\kappa = 12$, on top of relations $S^4 = (ST)^3 = 1$, one gets $T^{4\kappa=48} = 1$ (and $T^s \neq 1$ for smaller powers of T). The representation actually factorizes through a congruence subgroup of $SL(2,\mathbb{Z})$ and one obtains a representation of $SL(2,\mathbb{Z}/48\mathbb{Z})$ (one can check that all the defining relations given in [9] are verified).

2.2.7. Summary of notations

- G is the chosen Dynkin diagram of type ADE. It has r vertices. We also call G the fusion algebra (graph algebra) of this Dynkin diagram, when it exists.
- $\mathcal{G}$ is the adjacency matrix of G.
- κ is the Coxeter number of G.
- q is a primitive root of unity such that $q^{2\kappa} = 1$.
- $A_{\kappa-1}$ is the graph of type A with same Coxeter number κ as G.
- $N_i = (N_i)_k^j$ are the fusion matrices for the graph algebra (fusion algebra) of $A_{\kappa-1}$.
- $G_a = (G_a)_c^b$ are the fusion matrices for the graph algebra (fusion algebra) of G, when it exists.
- S_G is an $r \times r$ matrix that (in all cases but E_7 and D_{odd}) diagonalizes simultaneously the r fusion matrices G_a of the diagram G. When the diagram G is of type A, we just call it S.
- $E_a = (E_a)_b^i$ are the essential matrices for the graph G.
- $F_i = (F_i)_b^a \doteq (E_a)_b^i$ provide a representation of the graph algebra of $A_{\kappa-1}$. Matrices F_i (or E_a) describe the couplings between vertices a, b of G and the vertex i of $A_{\kappa-1}$.
- $\mathrm{Oc}(G)$ is the Ocneanu graph associated with G.
- $\Sigma_x = (\Sigma_x)_b^a$ are matrices describing the dual couplings between vertices a, b of G and the vertex x of the Ocneanu graph $\mathrm{Oc}(G)$.
- $W_x = (W_x)_j^i$ are the toric matrices (of size $(\kappa-1) \times (\kappa-1)$) associated with the vertices of $\mathrm{Oc}(G)$.
- Z_x is the twisted partition function associated with W_x.

[3] This is also discussed in a very recent preprint [5].



$$\mathcal{G}_{A_4} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Fig. 1. The A_4 Dynkin diagram and its adjacency matrix.

3. The A_n cases

3.1. A_4

The A_4 Dynkin diagram and its adjacency matrix are displayed in Fig. 1, where we use the following order for the basis: $\{\tau_0, \tau_1, \tau_2, \tau_3\}$.

Here $\kappa = 5$, the norm of the graph is the golden number $\beta = 2\cos(\pi/5) = (1+\sqrt{5})/2$, and the normalized Perron–Frobenius vector is $D = ([1]_q, [2]_q, [2]_q, [1]_q)$.

The A_4 Dynkin diagram determines in a unique way the graph algebra of A_4, whose multiplication table is displayed in Table 1.

The fusion matrices N_i are given by the following polynomials:

$$N_0 = Id_4 \text{ (the identity matrix)}, \qquad N_1 = \mathcal{G}_{A_4}, \qquad N_2 = N_1 \cdot N_1 - N_0,$$
$$N_3 = N_1 \cdot N_1 \cdot N_1 - 2 \cdot N_1.$$

They provide a faithful realization of the fusion algebra A_4. In the chosen basis, they read:

$$N_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad N_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$
$$N_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad N_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Table 1
Multiplication table for the A_4 graph algebra

	0	1	2	3
0	0	1	2	3
1	1	0 + 2	1 + 3	2
2	2	1 + 3	0 + 2	1
3	3	2	1	0

We form the tensor product $A_4 \otimes A_4$, whose dimension is 16, but we take it over A_4. The Ocneanu algebra of A_4 can be realized as the algebra of dimension 4 defined by

$$\mathcal{H}_{\mathrm{Oc}(A_4)} = A_4 \dot{\otimes} A_4 \doteq \frac{A_4 \otimes A_4}{A_4} = A_4 \otimes_{A_4} A_4.$$

It is spanned by a basis with four elements:

$$\underline{0} = 0\dot{\otimes}0, \qquad \underline{1} = 1\dot{\otimes}0, \qquad \underline{2} = 2\dot{\otimes}0, \qquad \underline{3} = 3\dot{\otimes}0,$$

and is isomorphic to the graph algebra A_4 itself. For this reason, the Ocneanu graph $\mathrm{Oc}(A_4)$ is the same as the Dynkin diagram A_4. Its elements are of the kind $m\dot{\otimes}n = 0\dot{\otimes}mn = mn\dot{\otimes}0$. The dimensions d_n, with n in $(0, 1, 2, 3)$, for the four blocks of the Racah–Wigner–Ocneanu bi-algebra $\mathcal{A}$ endowed with its first multiplicative law are respectively: $(4, 6, 6, 4)$. For its other multiplicative law (convolution), the dimensions d_x of the four blocks, labelled with x in the list $(0\dot{\otimes}0, 1\dot{\otimes}0, 2\dot{\otimes}0, 3\dot{\otimes}0)$ are also respectively: $(4, 6, 6, 4)$.

We have of course $\sum d_n = \sum d_x = 20$ and $\sum d_n^2 = \sum d_n^2 = 104$ but this observation is trivial in that case.

In the A_4 case (as in all A_n cases) the essential matrices E_i happen to be the same as the N_i matrices. The four toric matrices W_{ab} of the A_4 model are also equal to the N_i matrices, $W_{00} = N_0$ being the modular invariant. We write them as sesquilinear forms (the twisted partition functions given in the appendix).

3.2. A_n

We display in Fig. 2 the Dynkin diagram of A_n, for $n > 4$.

In all A_n cases, the graph algebra is completely determined, in a unique way, by the data of the corresponding Dynkin diagram. The Ocneanu algebra of A_n can be realized as

$$\mathcal{H}_{\mathrm{Oc}(A_n)} = A_n \dot{\otimes} A_n \doteq A_n \otimes_{A_n} A_n,$$

and appears to be isomorphic to the graph algebra A_n itself. Due to this fact, which occurs only in the A_n cases, the Ocneanu graphs are also equal to the corresponding Dynkin

$$\mathcal{G}_{A_n} = \begin{pmatrix} 0 & 1 & . & . & . & \cdots & . & . \\ 1 & 0 & 1 & . & . & \cdots & . & . \\ . & 1 & 0 & 1 & . & \cdots & . & . \\ . & . & 1 & 0 & 1 & \cdots & . & . \\ . & . & . & 1 & 0 & \cdots & . & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & . & . \\ . & . & . & . & . & . & . & 1 \\ . & . & . & . & . & . & 1 & 0 \end{pmatrix}$$

Fig. 2. The A_n Dynkin diagram and its adjacency matrix.

diagrams. The fusion matrices N_i are given by the following polynomials:

$$\begin{aligned} N_0 &= Id_n, \\ N_1 &= \mathcal{G}_{A_n}, \\ N_2 &= N_1 \cdot N_1 - N_0, \\ \vdots &= \vdots \\ N_i &= N_{i-1} \cdot N_1 - N_{i-2}. \end{aligned}$$

The essential matrices, as well as the n toric matrices of the A_n model are equal to these fusion matrices. We just give the modular invariant in sesquilinear form

$$A_n : Z_{\underline{0}} = \sum_{i=0}^{n} |\chi_n|^2 \quad \forall n \geq 3.$$

It is easy to see that, for A_n, the dimensions d_p of the blocks, for p from 0 to $n-1$ are given by $d_p = (p+1)(n-p)$.

4. The E_6 case

The E_6 diagram and its adjacency matrix are displayed in Fig. 3. We use the following order for the vertices: $\{\sigma_0, \sigma_1, \sigma_2, \sigma_5, \sigma_4, \sigma_3\}$.

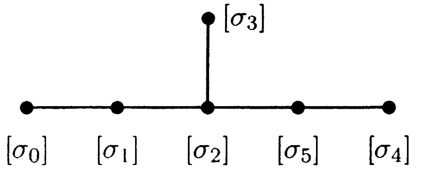
Here $\kappa = 12$, the norm of the graph is $\beta = 2\cos(\pi/12) = (1+\sqrt{3})/\sqrt{2}$ and the normalized Perron–Frobenius vector is $D = ([1]_q, [2]_q, [3]_q, [2]_q, [1]_q, [3]_q/[2]_q)$.

The E_6 Dynkin diagram determines in a unique way the multiplication table for the graph algebra of E_6, displayed in Table 2.

The fusion matrices G_i are given by the following polynomials:

$$\begin{aligned} &G_0 = Id_6, \qquad G_1 = \mathcal{G}_{E_6}, \qquad G_2 = G_1 \cdot G_1 - G_0, \\ &G_4 = G_1 \cdot G_1 \cdot G_1 \cdot G_1 - 4G_1 \cdot G_1 + 2G_0, \qquad G_5 = G_1 \cdot G_4, \\ &G_3 = -G_1 \cdot (G_4 - G_1 \cdot G_1 + 2G_0). \end{aligned}$$

Essential matrices have 6 columns and 11 rows. They are labelled by vertices of diagrams E_6 and A_{11}. They are calculated as explained in Section 2.2.3. With the order chosen for



$$\mathcal{G}_{E_6} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Fig. 3. The E_6 Dynkin diagram and its adjacency matrix.

Table 2
Multiplication table for the graph algebra of E_6

	0	1	2	5	4	3
0	0	1	2	5	4	3
1	1	$0+2$	$1+3+5$	$2+4$	5	2
2	2	$1+3+5$	$0+2+2+4$	$1+3+5$	2	$1+5$
5	5	$2+4$	$1+3+5$	$0+2$	1	2
4	4	5	2	1	0	3
3	3	2	$1+5$	2	3	$0+4$

vertices (012543) notice that the first row of matrix E_5, e.g., is $E_5(0) = (000100)$. The first essential matrix E_0 (essential paths leaving the origin) is given in Fig. 4, together with the corresponding induction–restriction graph (E_6 diagram with vertices labelled by A_{11} vertices).

The subspace A_3 generated by the elements $\{0, 3, 4\}$ is a subalgebra of the graph algebra of E_6 and leaves invariant (by multiplication) the complementary vector subspace generated by $\{1, 2, 5\}$. In other words, the subalgebra A_3 of E_6 admits a two-sided A_3-invariant complement. We form the tensor product $E_6 \otimes E_6$, but we take it over the subalgebra A_3 and define the following algebra:

$$\mathcal{H}_{\mathrm{Oc}(E_6)} = E_6 \dot{\otimes} E_6 \doteq \frac{E_6 \otimes E_6}{A_3} = E_6 \otimes_{A_3} E_6.$$

We have, e.g., $3\dot{\otimes}1 = 0\dot{\otimes}31 = 0\dot{\otimes}2$, and $4\dot{\otimes}1 = 0\dot{\otimes}41 = 0\dot{\otimes}5$.

$\mathcal{H}_{\mathrm{Oc}(E_6)}$ is spanned by a basis with 12 elements:

$$\begin{array}{llll} \underline{0} = 0\dot{\otimes}0, & \underline{3} = 3\dot{\otimes}0, & \underline{1'} = 0\dot{\otimes}1, & \underline{31'} = 3\dot{\otimes}1, \\ \underline{1} = 1\dot{\otimes}0, & \underline{4} = 4\dot{\otimes}0, & \underline{11'} = 1\dot{\otimes}1, & \underline{41'} = 4\dot{\otimes}1, \\ \underline{2} = 2\dot{\otimes}0, & \underline{5} = 5\dot{\otimes}0, & \underline{21'} = 2\dot{\otimes}1, & \underline{51'} = 5\dot{\otimes}1. \end{array}$$

$$E_0 = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix}$$

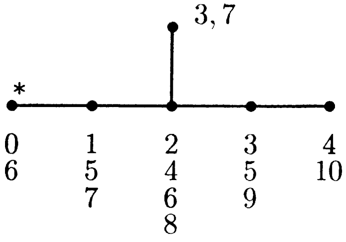


Fig. 4. Essential matrix E_0 and essential paths from the vertex 0 for the E_6 model.

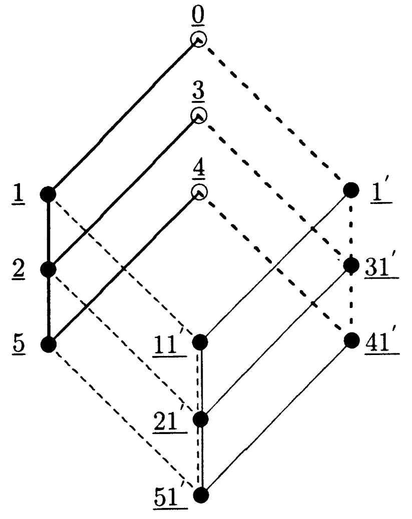


Fig. 5. The E_6 Ocneanu graph.

The element $0\dot{\otimes}0$ is the identity. The elements $1\dot{\otimes}0$ and $0\dot{\otimes}1$ are respectively the chiral left and right generators; they span separately two subalgebras $E_6 \otimes 0$ and $0 \otimes E_6$, both isomorphic with the graph algebra itself. The ambichiral part is the linear span of $\{\underline{0}, \underline{3}, \underline{4}\}$. We can easily check that multiplication by generators of $\mathcal{H}_{\mathrm{Oc}(E_6)}$ is indeed encoded by the Ocneanu graph of E_6, represented in Fig. 5. The full lines encode multiplication by the chiral left generator $\underline{1}$. For example: $\underline{1} \cdot \underline{2} = \underline{1} + \underline{3} + \underline{5}$ and in the E_6 Ocneanu graph the vertices $\underline{1}, \underline{3}$ and $\underline{5}$ are joined to the vertex $\underline{2}$ by a full line. The dashed lines encode multiplication by the chiral right generator $\underline{1'}$. For example: $\underline{1'} \cdot \underline{4} = \underline{41'}$ and in the E_6 Ocneanu graph the vertices $\underline{4}$ and $\underline{41'}$ are joined by a dashed line.

The dimensions d_n, with n in $(0, 1, 2, \ldots, 10)$, for the 11 blocks of the Racah–Wigner–Ocneanu bi-algebra $\mathcal{A}$ endowed with its first multiplicative law are respectively

$$(6, 10, 14, 18, 20, 20, 20, 18, 14, 10, 6).$$

For its other multiplicative law (convolution), the dimensions d_x of the 12 blocks, labelled with x in the list $(0\dot{\otimes}0, 3\dot{\otimes}0, 4\dot{\otimes}0, 1\dot{\otimes}0, 2\dot{\otimes}0, 5\dot{\otimes}0, 0\dot{\otimes}1, 0\dot{\otimes}2, 0\dot{\otimes}5, 1\dot{\otimes}1, 2\dot{\otimes}1, 5\dot{\otimes}1)$ are respectively

$$(6, 8, 6, 10, 14, 10, 10, 14, 10, 20, 28, 20)$$

Notice that $\sum d_n = \sum d_x = 156$ and $\sum d_n^2 = \sum d_x^2 = 2512$.

The 12 toric matrices W_{ab} of the E_6 model are obtained as explained in Section 2.2.5; for instance $W_{4\dot{\otimes}1} = E_4 \cdot \tilde{E}_1^{\mathrm{red}}$. We recall only the matrix expression of W_{00} (the modular invariant itself). The 11 other matrices,[4] are written as sesquilinear forms in the appendix

[4] They were already given in [6].

(they are the twisted partition functions).

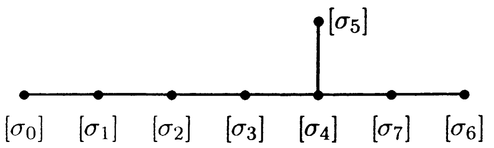
$$W_{00} = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}.$$

5. The E_8 case

The E_8 Dynkin diagram and its adjacency matrix are displayed in Fig. 6. We use the following order for the vertices: $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_7, \sigma_6, \sigma_5\}$.

Here $\kappa = 30$, the norm of the graph is $\beta = 2\cos(\pi/30)$ and the normalized Perron–Frobenius vector is $D = ([1]_q, [2]_q, [3]_q, [4]_q, [5]_q, [7]_q/[2]_q, [5]_q/[3]_q, [5]_q/[2]_q)$.

As for the E_6 case, the E_8 Dynkin diagram determines in a unique way the multiplication table for the graph algebra of E_8, displayed in Table 3.



$$\mathcal{G}_{E_8} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Fig. 6. The E_8 Dynkin diagram and its adjacency matrix.

Table 3
Multiplication table for the graph algebra of E_8.

	0	1	2	3	4	7	6	5
0	0	1	2	3	4	7	6	5
1	1	$0+2$	$1+3$	$2+4$	$3+5+7$	$4+6$	7	4
2	2	$1+3$	$0+2+4$	$1+3+5+7$	$2+4_2+6$	$3+5+7$	4	$3+7$
3	3	$2+4$	$1+3+5+7$	$0+2+4_2+6$	$1+3_2+5+7_2$	$2+4_2$	$3+5$	$2+4+6$
4	4	$3+5+7$	$2+4_2+6$	$1+3_2+5+7_2$	$0+2_2+4_3+6$	$1+3_2+5+7$	$2+4$	$1+3+5+7$
7	7	$4+6$	$3+5+7$	$2+4+4$	$1+3_2+5+7$	$0+2+4+6$	$1+7$	$2+4$
6	6	7	4	$3+5$	$2+4$	$1+7$	$0+6$	3
5	5	4	$3+7$	$2+4+6$	$1+3+5+7$	$2+4$	3	$0+4$

$$E_0 = \begin{pmatrix}
1 & . & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . \\
. & . & . & . & 1 & . & . & . \\
. & . & . & . & . & 1 & . & 1 \\
. & . & . & . & 1 & . & 1 & . \\
. & . & . & 1 & . & 1 & . & . \\
. & . & 1 & . & 1 & . & . & . \\
. & 1 & . & 1 & . & . & . & 1 \\
1 & . & 1 & . & 1 & . & . & . \\
. & 1 & . & 1 & . & 1 & . & . \\
. & . & 1 & . & 1 & . & 1 & . \\
. & . & . & 1 & . & 1 & . & 1 \\
. & . & . & . & 2 & . & . & . \\
. & . & . & 1 & . & 1 & . & 1 \\
. & . & 1 & . & 1 & . & 1 & . \\
. & 1 & . & 1 & . & 1 & . & . \\
1 & . & 1 & . & 1 & . & . & . \\
. & 1 & . & 1 & . & . & . & 1 \\
. & . & 1 & . & 1 & . & . & . \\
. & . & . & 1 & . & 1 & . & . \\
. & . & . & . & 1 & . & 1 & . \\
. & . & . & . & . & 1 & . & 1 \\
. & . & . & . & 1 & . & . & . \\
. & . & . & 1 & . & . & . & . \\
. & . & 1 & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . \\
1 & . & . & . & . & . & . & .
\end{pmatrix}$$

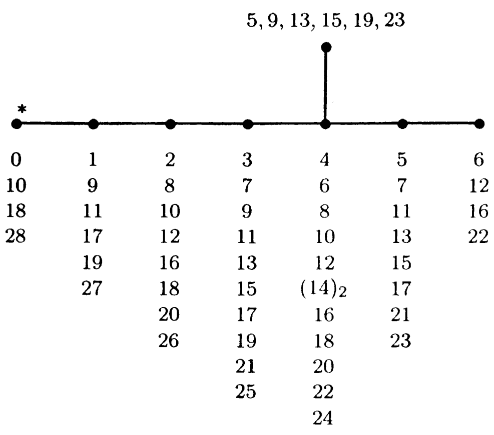


Fig. 7. Essential matrix E_0 and essential paths from the vertex 0 for the E_8 model.

The fusion matrices G_i are given by the following polynomials:

$$G_0 = Id_8, \qquad G_1 = \mathcal{G}_{E_8}, \qquad G_2 = G_1 \cdot G_1 - G_0, \qquad G_3 = G_1 \cdot G_1 \cdot G_1 - 2G_1,$$
$$G_4 = G_1 \cdot G_1 \cdot G_1 \cdot G_1 - 3G_1 \cdot G_1 + G_0, \qquad G_6 = G_2 \cdot G_4 - G_2 - 2G_4,$$
$$G_7 = G_1 \cdot G_6, \qquad G_5 = G_2 \cdot G_7 - G_3 - G_7.$$

Essential matrices have 8 columns and 29 rows. They are labelled by vertices of diagrams E_8 and A_{29}. The first essential matrix E_0 (essential paths leaving the origin) is given in Fig. 7, together with the corresponding induction–restriction graph (E_8 diagram with vertices labelled by A_{29} vertices).

The subspace A_2 generated by the elements $\{0, 6\}$ is a subalgebra of the graph algebra of E_8 that admits a two-sided A_2-invariant complement. We form the tensor product $E_8 \otimes E_8$, but we take it over the subalgebra A_2. The Ocneanu algebra of E_8 can be realized as

$$\mathcal{H}_{\mathrm{Oc}(E_8)} = E_8 \dot{\otimes} E_8 \doteq \frac{E_8 \otimes E_8}{A_2} = E_8 \otimes_{A_2} E_8.$$

For instance $6\dot{\otimes}0 = 0\dot{\otimes}6$, $6\dot{\otimes}1 = 0\dot{\otimes}7$, $6\dot{\otimes}2 = 0\dot{\otimes}4$, $6\dot{\otimes}5 = 0\dot{\otimes}3$.

$\mathcal{H}_{\mathrm{Oc}(E_8)}$ is spanned by a basis with 32 elements:

$$
\begin{array}{llll}
\underline{0} = 0\dot{\otimes}0, & \underline{1'} = 0\dot{\otimes}1, & \underline{2'} = 0\dot{\otimes}2, & \underline{5'} = 0\dot{\otimes}5, \\
\underline{1} = 1\dot{\otimes}0, & \underline{11'} = 1\dot{\otimes}1, & \underline{12'} = 1\dot{\otimes}2, & \underline{15'} = 1\dot{\otimes}5, \\
\underline{2} = 2\dot{\otimes}0, & \underline{21'} = 2\dot{\otimes}1, & \underline{22'} = 2\dot{\otimes}2, & \underline{25'} = 2\dot{\otimes}5, \\
\underline{3} = 3\dot{\otimes}0, & \underline{31'} = 3\dot{\otimes}1, & \underline{32'} = 3\dot{\otimes}2, & \underline{35'} = 3\dot{\otimes}5, \\
\underline{4} = 4\dot{\otimes}0, & \underline{41'} = 4\dot{\otimes}1, & \underline{42'} = 4\dot{\otimes}2, & \underline{45'} = 4\dot{\otimes}5, \\
\underline{5} = 5\dot{\otimes}0, & \underline{51'} = 5\dot{\otimes}1, & \underline{52'} = 5\dot{\otimes}2, & \underline{55'} = 5\dot{\otimes}5, \\
\underline{6} = 6\dot{\otimes}0, & \underline{61'} = 6\dot{\otimes}1, & \underline{62'} = 6\dot{\otimes}2, & \underline{65'} = 6\dot{\otimes}5, \\
\underline{7} = 7\dot{\otimes}0, & \underline{71'} = 7\dot{\otimes}1, & \underline{72'} = 7\dot{\otimes}2, & \underline{75'} = 7\dot{\otimes}5.
\end{array}
$$

The element $0\dot{\otimes}0$ is the identity. The elements $1\dot{\otimes}0$ and $0\dot{\otimes}1$ are respectively the chiral left and right generators; they span independently the subalgebras $E_8 \otimes 0$ and $0 \otimes E_8$. One can easily check that multiplication by these two generators is indeed encoded by the Ocneanu graph of E_8, represented in Fig. 8. Full lines (respectively dashed lines) encode multiplication by the chiral left (respectively chiral right) generator. The ambichiral part is the linear span of $\{\underline{0}, \underline{6}\}$.

The dimensions d_n, with n in $(0, 1, 2, \ldots, 28)$, for the 29 blocks of the Racah–Wigner–Ocneanu bi-algebra $\mathcal{A}$ endowed with its first multiplicative law are respectively

$$(8, 14, 20, 26, 32, 38, 44, 48, 52, 56, 60, 62, 64, 64, 64, 64, 64, 62, 60, 56, 52, 48, 44, 38, 32, 26, 20, 14, 8).$$

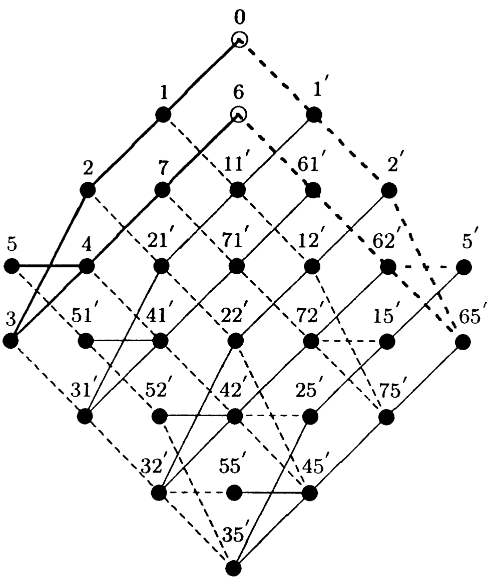


Fig. 8. The E_8 Ocneanu graph.

For its other multiplicative law (convolution), the dimensions d_x of the 32 blocks, labelled with x in the list ($0\dot{\otimes}0$, $1\dot{\otimes}0$, $2\dot{\otimes}0$, $3\dot{\otimes}0$, $4\dot{\otimes}0$, $5\dot{\otimes}0$, $6\dot{\otimes}0$, $7\dot{\otimes}0$, $0\dot{\otimes}1$, $1\dot{\otimes}1$, $2\dot{\otimes}1$, $3\dot{\otimes}1$, $4\dot{\otimes}1$, $5\dot{\otimes}1$, $6\dot{\otimes}1$, $7\dot{\otimes}1$, $0\dot{\otimes}2$, $1\dot{\otimes}2$, $2\dot{\otimes}2$, $3\dot{\otimes}2$, $4\dot{\otimes}2$, $5\dot{\otimes}2$, $6\dot{\otimes}2$, $7\dot{\otimes}2$, $0\dot{\otimes}5$, $1\dot{\otimes}5$, $2\dot{\otimes}5$, $3\dot{\otimes}5$, $4\dot{\otimes}5$, $5\dot{\otimes}5$, $6\dot{\otimes}5$, $7\dot{\otimes}5$) are respectively given by (8, 14, 20, 26, 32, 16, 12, 22, 14, 28, 40, 52, 64, 32, 22, 44, 20, 40, 60, 78, 96, 48, 32, 64, 16, 32, 48, 64, 78, 40, 26, 52).

Notice that $\sum d_n = \sum d_x = 1240$ and $\sum d_n^2 = \sum d_x^2 = 63136$.

The 32 toric matrices W_{ab} of the E_8 model are obtained as explained in Section 2.2.5; for instance $W_{\underline{52'}} = W_{5\dot{\otimes}2} = E_5 \cdot \tilde{E}_2^{\text{red}}$. We recall only the matrix expression of the modular invariant W_{00}. The partition functions corresponding to all the toric matrices are given as sesquilinear forms in the appendix.

$$W_{00} = \begin{pmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
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\end{pmatrix}.$$

6. The D_{even} case

General formulae valid for all cases of this family are a bit heavy. We therefore only provide a detailed treatment of the cases D_4 and D_6 but generalization is straightforward.

6.1. The D_4 case

The D_4 diagram and its adjacency matrix are displayed in Fig. 9. We use the following order for the vertices: $\{\sigma_0, \sigma_1, \sigma_2, \sigma_{2'}\}$.

Here $\kappa = 6$, the norm of the graph is $\beta = 2\cos(\pi/6) = \sqrt{3}$ and the normalized Perron–Frobenius vector is $D = ([1]_q, [2]_q, [2]_q/[2]_q = 1, [2]_q/[2]_q = 1)$.

For the D_4 case (as for all the D_{2n} cases), we have to impose that the structure constants of its graph algebra should be positive integers, in order for the Dynkin diagram to determine in a unique way the multiplication table of the graph algebra, (displayed in Table 4).

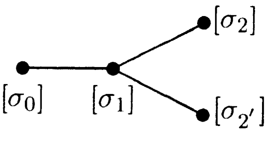
The fusion matrices G_i are given by the following polynomials:

$$G_0 = Id_4, \qquad G_1 = \mathcal{G}_{D_4}, \qquad G_2 + G_{2'} = G_1 \cdot G_1 - G_0.$$

Imposing that entries of G_2 and $G_{2'}$ should be positive integers leads to a unique solution (up to $G_2 \leftrightarrow G_{2'}$), namely:

$$G_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad G_2' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Essential matrices have 4 columns and 5 rows. They are labelled by vertices of diagrams D_4 and A_5. The first essential matrix E_0 is given in Fig. 10, together with the corresponding induction–restriction graph (D_4 diagram with vertices labelled by A_5 vertices).



$$\mathcal{G}_{D_4} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Fig. 9. The D_4 Dynkin diagram and its adjacency matrix.

Table 4
Multiplication table for the graph algebra of D_4

	0	1	2	2'
0	0	1	2	2'
1	1	0 + 2 + 2'	1	1
2	2	1	2'	0
2'	2'	1	0	2

$$E_0 = \begin{pmatrix} 1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 & 1 \\ . & 1 & . & . \\ 1 & . & . & . \end{pmatrix}$$

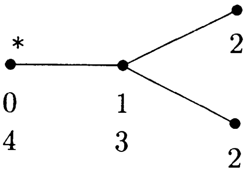


Fig. 10. Essential matrix E_0 and essential paths from the vertex 0 for the D_4 model.

The subspace J_3 generated by the elements $\{0, 2, 2'\}$, is a subalgebra of the graph algebra of D_4 that admits a two-sided J_3-invariant complement. We first form the tensor product $D_4 \otimes D_4$, but we take it over the subalgebra J_3. We get the algebra $D_4^{\dot\otimes} = D_4 \dot\otimes D_4 \dot= D_4 \otimes_{J_3} D_4$, spanned by a basis with six elements

$$0\dot\otimes 0, \qquad 1\dot\otimes 0, \qquad 2\dot\otimes 0, \qquad 2'\dot\otimes 0, \qquad 0\dot\otimes 1, \qquad 1\dot\otimes 1.$$

The Ocneanu algebra of D_4, $\mathcal{H}_{\mathrm{Oc}(D_4)}$, can be realized as a subalgebra of dimension 8 of the following non-commutative algebra:

$$D_4^{\dot\otimes} \oplus M(2, \mathbb{C}).$$

The eight elements of the basis are given by

$$\underline{0} = 0\dot\otimes 0 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \underline{\epsilon} = \tfrac{1}{3}(1\dot\otimes 1) + \theta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\underline{1} = 1\dot\otimes 0 + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \underline{1\epsilon} = 0\dot\otimes 1 + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\underline{2} = 2\dot\otimes 0 + \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \qquad \underline{2\epsilon} = \tfrac{1}{3}(1\dot\otimes 1) + \theta \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix},$$

$$\underline{2'} = 2'\dot\otimes 0 + \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}, \qquad \underline{2'\epsilon} = \tfrac{1}{3}(1\dot\otimes 1) + \theta \begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix}.$$

where

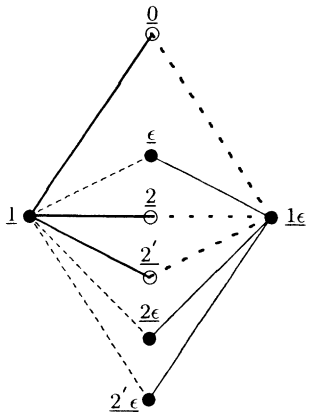
$$\theta^2 = 0 \ \text{(Grassmann parameter)}, \qquad \alpha = \frac{-1 + \mathrm{i}\sqrt{3}}{2}, \qquad \beta = \frac{-1 - \mathrm{i}\sqrt{3}}{2}.$$

The multiplication in this algebra is defined by

$$((e_1 \dot\otimes f_1) + A) \cdot ((e_2 \dot\otimes f_2) + B) = (e_1 \cdot e_2)\dot\otimes(f_1 \cdot f_2) + A \cdot B,$$

where $e_1, f_1, e_2, f_2 \in D_4^{\dot\otimes}$ and $A, B \in M(2, \mathbb{C})$.

The numbers α and β are determined by the multiplication table of $\mathcal{H}_{\mathrm{Oc}(D_4)}$. For example, the relations $\underline{1} \cdot \underline{1} = \underline{0} + \underline{2} + \underline{2'}$, $\underline{2} \cdot \underline{2} = \underline{2'}$ and $\underline{2} \cdot \underline{2'} = \underline{0}$ lead to the equations: $\alpha + \beta = -1$, $\alpha \cdot \beta = 1$, $\alpha^2 = \beta$ and $\beta^2 = \alpha$, that determines uniquely α and β.



$$W_{00} = \begin{pmatrix} 1 & . & . & . & 1 \\ . & . & . & . & . \\ . & . & 2 & . & . \\ . & . & . & . & . \\ 1 & . & . & . & 1 \end{pmatrix}$$

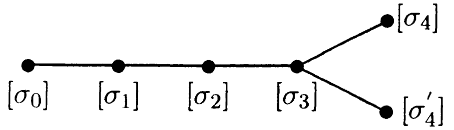
Fig. 11. The D_4 Ocneanu graph and the modular invariant matrix.

Sketch of our construction. We first define $D_4^{\dot{\otimes}}$ by quotienting the tensor square of D_4 by the subalgebra J_3 that admits a two-sided J_3-invariant complement. From the graph $\mathrm{Oc}(D_4)$ taken from [16], we see that $\underline{1}$ and $\underline{1\epsilon}$ separately generate the left and right subalgebras isomorphic with the graph algebra of D_4, therefore we set $\underline{1} = 1\dot{\otimes}0$ and $\underline{1\epsilon} = 0\dot{\otimes}1$. We also see that $\underline{1} \cdot \underline{\epsilon} = \underline{1\epsilon}$; this equality implies that the $D_4^{\dot{\otimes}}$ part of $\underline{\epsilon}$ should be proportional to $1\dot{\otimes}1$ since $(1\dot{\otimes}0)(1\dot{\otimes}1) = (0 + 2 + 2')\dot{\otimes}1 = 3(0\dot{\otimes}1)$. The matrix part of ϵ and of the other generators (the coefficients α and β) can then be determined by imposing that the obtained multiplication table should coincide with the multiplication table constructed from the Ocneanu graph $\mathrm{Oc}(D_4)$. Such a construction can be generalized to all D_{even} cases.

The element $\underline{0}$ is the identity. The elements $\underline{1}$ and $\underline{1\epsilon}$ are respectively the chiral left and right generators. The multiplication table of this algebra is given in Table 5, and we can check that multiplication by the generators is indeed encoded by the Ocneanu graph of D_4, represented in Fig. 11. Warning: the table is not symmetric (the multiplication is not commutative); for instance $\underline{2\epsilon} = \underline{2} \cdot \underline{\epsilon} \neq \underline{\epsilon} \cdot \underline{2}$. The ambichiral part is the linear span of $\{\underline{0}, \underline{2}, \underline{2'}\}$.

Table 5
Multiplication table of the Ocneanu algebra of D_4

	$\underline{0}$	$\underline{1}$	$\underline{1\epsilon}$	$\underline{\epsilon}$	$\underline{2}$	$\underline{2'}$	$\underline{2\epsilon}$	$\underline{2'\epsilon}$
$\underline{0}$	$\underline{0}$	$\underline{1}$	$\underline{1\epsilon}$	$\underline{\epsilon}$	$\underline{2}$	$\underline{2'}$	$\underline{2\epsilon}$	$\underline{2'\epsilon}$
$\underline{1}$	$\underline{1}$	$\underline{0} + \underline{2} + \underline{2'}$	$\underline{\epsilon} + \underline{2\epsilon} + \underline{2'\epsilon}$	$\underline{1\epsilon}$	$\underline{1}$	$\underline{1}$	$\underline{1\epsilon}$	$\underline{1\epsilon}$
$\underline{1\epsilon}$	$\underline{1\epsilon}$	$\underline{\epsilon} + \underline{2\epsilon} + \underline{2'\epsilon}$	$\underline{0} + \underline{2} + \underline{2'}$	$\underline{1}$	$\underline{1\epsilon}$	$\underline{1\epsilon}$	$\underline{1}$	$\underline{1}$
$\underline{\epsilon}$	$\underline{\epsilon}$	$\underline{1\epsilon}$	$\underline{1}$	$\underline{\eta}$	$\underline{2'\epsilon}$	$\underline{2\epsilon}$	$\underline{2'\eta}$	$\underline{2\eta}$
$\underline{2}$	$\underline{2}$	$\underline{1}$	$\underline{1\epsilon}$	$\underline{2\epsilon}$	$\underline{2'}$	$\underline{0}$	$\underline{2'\epsilon}$	$\underline{\epsilon}$
$\underline{2'}$	$\underline{2'}$	$\underline{1}$	$\underline{1\epsilon}$	$\underline{2'\epsilon}$	$\underline{0}$	$\underline{2}$	$\underline{\epsilon}$	$\underline{2\epsilon}$
$\underline{2\epsilon}$	$\underline{2\epsilon}$	$\underline{1\epsilon}$	$\underline{1}$	$\underline{2\eta}$	$\underline{\epsilon}$	$\underline{2'\epsilon}$	$\underline{\eta}$	$\underline{2'\eta}$
$\underline{2'\epsilon}$	$\underline{2'\epsilon}$	$\underline{1\epsilon}$	$\underline{1}$	$\underline{2'\eta}$	$\underline{2\epsilon}$	$\underline{\epsilon}$	$\underline{2\eta}$	$\underline{\eta}$



$$\mathcal{G}_{D_6} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Fig. 12. The D_6 Dynkin diagram and its adjacency matrix.

Table 6
Multiplication table for the graph algebra of D_6

	0	1	2	3	4	$4'$
0	0	1	2	3	4	$4'$
1	1	$0+2$	$1+3$	$2+4+4'$	3	3
2	2	$1+3$	$0+2+4+4'$	$1+3+3$	$2+4'$	$2+4$
3	3	$2+4+4'$	$1+3+3$	$0+2+2+4+4'$	$1+3$	$1+3$
4	4	3	$2+4'$	$1+3$	$0+4$	2
$4'$	$4'$	3	$2+4$	$1+3$	2	$0+4'$

The dimensions d_n, with $n = 0, 1, 2, 3, 4$ are respectively $(4, 6, 8, 6, 4)$. We find that $\sum d_n = 28$ and $\sum d_n^2 = 168$.

The eight toric matrices W_{ab} of the D_4 model and the corresponding partition functions are obtained as usual. For instance $W_{\underline{\epsilon}} = W_{\underline{2\epsilon}} = W_{\underline{2'\epsilon}} = 1/3 E_1 \tilde{E}_1^{\mathrm{red}}$. We recall the matrix expression of the modular invariant W_{00} and give the others toric matrices as sesquilinear forms in the appendix. [5]

6.2. The D_6 case

The D_6 Dynkin diagram and its adjacency matrix are displayed in Fig. 12. We use the following order for the vertices: $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_{4'}\}$.

Here $\kappa = 10$, the norm of the graph is $\beta = [2]_q = 2\cos(\pi/10) = ((5+\sqrt{5})/2)^{1/2}$ and the normalized Perron–Frobenius vector is $D = ([1]_q, [2]_q, [3]_q, [4]_q, [4]_q/[2]_q, [4]_q/[2]_q)$.

Imposing positivity, the table of multiplication of the graph algebra of D_6 is completely determined by its Dynkin diagram (Table 6).

The fusion matrices G_i are given by the following polynomials:

$$G_0 = Id_6, \qquad G_1 = \mathcal{G}_{D_6}, \qquad G_2 = G_1 \cdot G_1 - G_0, \qquad G_3 = G_2 \cdot G_1 - G_1,$$
$$G_4 + G_{4'} = G_1 \cdot G_3 - G_2$$

[5] The toric matrices of D_4 were already published in [21].

$$E_0 = \begin{pmatrix} 1 & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & 1 & . & . & . \\ . & . & . & 1 & . & . \\ . & . & . & . & 1 & 1 \\ . & . & . & 1 & . & . \\ . & . & 1 & . & . & . \\ . & 1 & . & . & . & . \\ 1 & . & . & . & . & . \end{pmatrix}$$

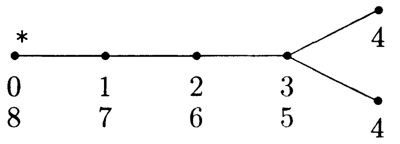


Fig. 13. Essential matrix E_0 and essential paths from the vertex 0 for the D_6 model.

Imposing that entries of G_4 and $G_{4'}$ should be positive integers leads to a unique solution (up to $G_4 \leftrightarrow G_{4'}$), namely:

$$G_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad G'_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Essential matrices have 6 columns and 9 rows. They are labelled by vertices of diagrams D_6 and A_9. The first essential matrix E_0 is given in Fig. 13, together with the corresponding induction–restriction graph (D_6 diagram with vertices labelled by A_9 vertices).

The subspace J_4 generated by the elements $\{0, 2, 4, 4'\}$ is a subalgebra of the graph algebra of D_6 that admits a two-sided J_4-invariant complement. We first form the tensor product $D_6 \otimes D_6$, but we take it over the subalgebra J_4. We get the algebra $D_6^{\dot\otimes} = D_6 \dot\otimes D_6 \doteq D_6 \otimes_{J_4} D_6$, spanned by a basis with 10 elements

$$\begin{array}{lllll} 0\dot\otimes 0, & 1\dot\otimes 0, & 2\dot\otimes 0, & 3\dot\otimes 0, & 4\dot\otimes 0, \\ 4'\dot\otimes 0, & 0\dot\otimes 1, & 0\dot\otimes 3, & 1\dot\otimes 1, & 1\dot\otimes 3. \end{array}$$

The Ocneanu algebra of D_6, $\mathcal{H}_{\mathrm{Oc}(D_6)}$, can be realized as a subalgebra of dimension 12 of the following non-commutative algebra:

$$D_6^{\dot\otimes} \oplus M(2, \mathbb{C}).$$

The 12 elements of the basis are given by

$$\underline{0} = 0\dot{\otimes}0 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \underline{\epsilon} = \tfrac{3}{5}(1\dot{\otimes}1) - \tfrac{1}{5}(3\dot{\otimes}1) + \theta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\underline{1} = 1\dot{\otimes}0 + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \underline{1\epsilon} = 0\dot{\otimes}1 + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\underline{2} = 2\dot{\otimes}0 + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \underline{2\epsilon} = \tfrac{2}{5}(1\dot{\otimes}1) + \tfrac{1}{5}(3\dot{\otimes}1) + \theta \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

$$\underline{3} = 3\dot{\otimes}0 + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \underline{3\epsilon} = 0\dot{\otimes}3 + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\underline{4} = 4\dot{\otimes}0 + \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \qquad \underline{4\epsilon} = -\tfrac{1}{5}(1\dot{\otimes}1) + \tfrac{2}{5}(3\dot{\otimes}1) + \theta \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix},$$

$$\underline{4'} = 4'\dot{\otimes}0 + \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}, \qquad \underline{4'\epsilon} = -\tfrac{1}{5}(1\dot{\otimes}1) + \tfrac{2}{5}(3\dot{\otimes}1) + \theta \begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix}.$$

where

$$\theta^2 = 0, \qquad \alpha = \frac{-1+\sqrt{5}}{2}, \qquad \beta = \frac{-1-\sqrt{5}}{2}.$$

The element $\underline{0}$ is the identity. The elements $\underline{1}$ and $\underline{1\epsilon}$ are respectively the chiral left and right generators. The multiplication by these generators is encoded by the Ocneanu graph of D_6, represented in Fig. 14. The ambichiral part is the linear span of $\{\underline{0}, \underline{2}, \underline{4}, \underline{4'}\}$.

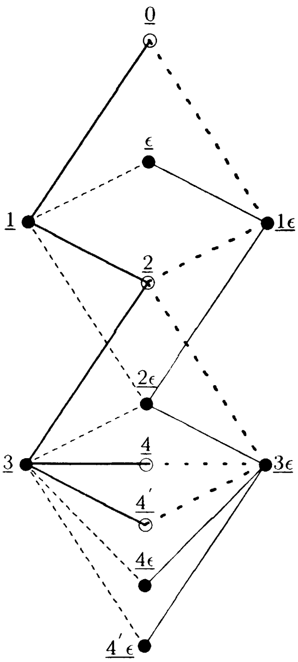
The dimensions d_n, with $n = 0, 1, 2, \ldots, 8$, are respectively (6, 10, 14, 16, 18, 16, 14, 10, 6). Therefore, $\sum d_n = 110$ and $\sum d_n^2 = 1500$.

The 12 toric matrices W_{ab} of the D_6 model and the corresponding partition functions are obtained as usual. For instance $W_{\underline{2\epsilon}} = (2/5)E_1\tilde{E}_1^{\mathrm{red}} + (1/5)E_3\tilde{E}_1^{\mathrm{red}}$. We recall the matrix expression of the modular invariant W_{00} and give the others as sesquilinears forms in the appendix.

6.3. *The D_{even} case*

In the case of D_{2s}, we first build $D_{2s}^{\dot{\otimes}} = D_{2s} \otimes D_{2s}/J_{s+1}$, of dimension $4s-2$ by dividing the tensor square of D_{2s} by the two-sided ideal generated by $u \otimes 0 - 0 \otimes u$, where u belongs to the subalgebra J_{s+1} spanned by $\{0, 2, 4, 6, \ldots, (2s-4), (2s-2), (2s-2)'\}$. This subalgebra admits a two-sided J_{s+1}-invariant complement. We then define $\mathcal{H}_{\mathrm{Oc}(D_{2s})}$ as a subalgebra of dimension $4s$ of $D_{2s}^{\dot{\otimes}} \oplus M(2, \mathbb{C})$. It is enough to know $\underline{0}$, $\underline{1}$ and $\underline{\epsilon}$ to build explicitly an algebra $\mathcal{H}_{\mathrm{Oc}(D_{2s})}$ from the graph $\mathrm{Oc}(D_{2s})$. We fix

$$\underline{0} = 0\dot{\otimes}0 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \underline{1} = 1\dot{\otimes}0 + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \underline{1\epsilon} = 0\dot{\otimes}1 + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$



$$W_{00}=\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

Fig. 14. The D_6 Ocneanu graph and the modular invariant matrix.

and set

$$\underline{\epsilon}=\sum a_\alpha \alpha \dot{\otimes} 1+\theta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\alpha \in \{1, 3, \ldots, 2s-3\}$ and where the a_α are scalars uniquely determined by the (linear) equation $\underline{1}\cdot\underline{\epsilon}=\underline{1\epsilon}$. The $D_{2s}^{\dot{\otimes}}$ parts of the other elements are then uniquely fixed. For the elements $(\underline{2}, \underline{3}, \ldots, \underline{(2s-2)}, \underline{(2s-2)'})$, it is $(2\dot{\otimes}0, 3\dot{\otimes}0, \ldots)$.

We write the matrix part of $\underline{(2s-2)}$, $\underline{(2s-2)'}$ as

$$\underline{(2s-2)}=\cdots+\theta\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \qquad \underline{(2s-2)'}=\cdots+\theta\begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}.$$

By imposing that the obtained multiplication table should coincide with the multiplication table constructed from the Ocneanu graph $\mathrm{Oc}(D_{2s})$, we determine uniquely the expression of the others elements, and find also the values of θ, α and β. In every case $\theta^2=0$. For α and β, we find

- s even: α and β are complexes

$$\alpha=\frac{-1+\mathrm{i}\sqrt{(2s-1)}}{2}, \qquad \beta=\frac{-1-\mathrm{i}\sqrt{(2s-1)}}{2}=\overline{\alpha}.$$

- s odd: α and β are reals

$$\alpha = \frac{-1+\sqrt{(2s-1)}}{2}, \qquad \beta = \frac{-1-\sqrt{(2s-1)}}{2}.$$

The tables of fusion for the cases s even and s odd have also a different structure, as it is clear from the examples D_4 and D_6 given in the previous sections.

7. The D_{odd} case

General formulae valid for all cases of this family are a bit heavy. We therefore only provide a detailed treatment of the cases D_5 and D_7 but generalization is straightforward.

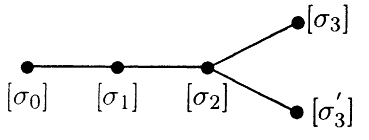
7.1. The D_5 case

The D_5 Dynkin diagram and its adjacency matrix are displayed in Fig. 15. We use the following order for the vertices: $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_{3'}\}$.

Here $\kappa = 8$, the norm of the graph is $\beta = [2]_q = 2\cos(\pi/8) = (2+\sqrt{2})^{1/2}$ and the normalized Perron–Frobenius vector is $D = ([1]_q, [2]_q, [3]_q, [3]_q/[2]_q, [3]_q/[2]_q)$.

In the D_5 case, as in all D_{odd} cases, it is not possible to define a graph algebra at all.

Essential matrices of D_5 have 5 columns and 7 rows. They are labelled by vertices of diagrams D_5 and A_7. The first essential matrix E_0 is given in Fig. 16, together with the corresponding induction–restriction graph (D_5 diagram with vertices labelled by A_7 vertices).



$$\mathcal{G}_{D_5} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Fig. 15. The D_5 Dynkin diagram and its adjacency matrix.

$$E_0 = \begin{pmatrix} 1 & . & . & . & . \\ . & 1 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & 1 \\ . & . & 1 & . & . \\ . & 1 & . & . & . \\ 1 & . & . & . & . \end{pmatrix}$$

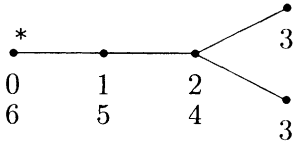


Fig. 16. Essential matrix E_0 and essential paths from the vertex 0 for the D_5 model.

The Ocneanu algebra of D_5 can be realized by using the graph algebra of A_7. For D_{2n+1}, we have to use the graph algebra of A_{4n-1}.

We form the tensor product $A_7 \otimes A_7$, and define an application $\rho : A_7 \to A_7$ such that

$$\rho(i) = i \quad \text{for } i \in \{0, 2, 3, 4, 6, 7\}, \qquad \rho(1) = 5, \quad \rho(5) = 1.$$

We take the tensor product over ρ, and define the Ocneanu algebra of D_5 as

$$\mathcal{H}_{\mathrm{Oc}(D_5)} = A_7 \dot{\otimes} A_7 \doteq \frac{A_7 \otimes A_7}{\rho(A_7)}.$$

For instance $2\dot{\otimes}0 = 0\dot{\otimes}\rho(2) = 0\dot{\otimes}2$, and $1\dot{\otimes}0 = 0\dot{\otimes}\rho(1) = 0\dot{\otimes}5$.

$\mathcal{H}_{\mathrm{Oc}(D_5)}$ is spanned by a basis with seven elements

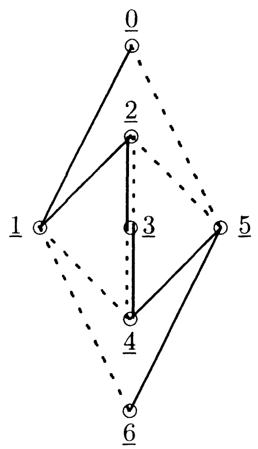
$$\underline{0} = 0\dot{\otimes}0, \qquad \underline{1} = 1\dot{\otimes}0 = 0\dot{\otimes}5, \qquad \underline{2} = 2\dot{\otimes}0 = 0\dot{\otimes}2, \qquad \underline{3} = 3\dot{\otimes}0 = 0\dot{\otimes}3,$$
$$\underline{4} = 4\dot{\otimes}0 = 0\dot{\otimes}4, \qquad \underline{5} = 5\dot{\otimes}0 = 0\dot{\otimes}1, \qquad \underline{6} = 6\dot{\otimes}0 = 0\dot{\otimes}6.$$

$1\dot{\otimes}0$ and $0\dot{\otimes}1$ are respectively the chiral left and right generators. The multiplication by these generators is encoded by the Ocneanu graph of D_5, represented in Fig. 17. All the points are ambichiral.

To obtain the toric matrices of the D_5 model, we need the essential matrices $E_i(A_7)$ of the A_7 case (we recall that in the A_n cases, the essential matrices are equal to the fusion matrices N_i). We define new essential matrices $E_i^{\rho}(A_7)$ by permuting the columns of $E_i(A_7)$ associated with the vertices 1 and 5. The toric matrices of the D_5 model are then obtained by setting

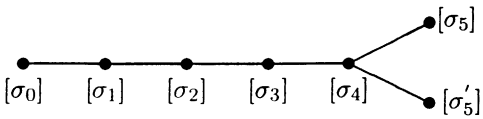
$$W[a, b] \doteq \cdot E_a(A_7) \cdot (\widetilde{E_b^{\rho}(A_7)}).$$

We recall the matrix expression of the modular invariant W_{00} and give the others as sesquilinears forms in the appendix.



$$W_{00} = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

Fig. 17. The D_5 Ocneanu graph and the modular invariant matrix.



$$\mathcal{G}_{D_7} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Fig. 18. The D_7 Dynkin diagram and its adjacency matrix.

7.2. *The D_7 case*

The D_7 Dynkin diagram and its adjacency matrix are displayed in Fig. 18. We use the following order for the vertices: $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_{5'}\}$.

Here $\kappa = 12$, the norm of the graph is $\beta = [2]_q = 2\cos(\pi/12) = (1+\sqrt{3})/\sqrt{2}$ and the normalized Perron–Frobenius vector is $D = ([1]_q, [2]_q, [3]_q, [4]_q, [5]_q, [5]_q/[2]_q, [5]_q/[2]_q)$.

Essential matrices have 7 columns and 11 rows. They are labelled by vertices of diagrams D_7 and A_{11}. The first essential matrix E_0 is given in Fig. 19, together with the corresponding induction–restriction graph (D_7 diagram with vertices labelled by A_{11} vertices).

The Ocneanu algebra of D_7 can be realized by using the graph algebra of A_{11}. We form the tensor product $A_{11} \otimes A_{11}$, and define an application $\rho : A_{11} \to A_{11}$ such that

$$\rho(i) = i \quad \text{for} \quad i \in \{0, 2, 4, 5, 6, 8, 10\}, \qquad \rho(1) = 9, \quad \rho(3) = 7,$$
$$\rho(7) = 3, \quad \rho(9) = 1.$$

We take the tensor product over ρ, and define the Ocneanu algebra of D_7 as

$$\mathcal{H}_{\mathrm{Oc}(D_7)} = A_{11} \dot{\otimes} A_{11} \doteq \frac{A_{11} \otimes A_{11}}{\rho(A_{11})}.$$

$$E_0 = \begin{pmatrix} 1 & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ . & . & 1 & . & . & . & . \\ . & . & . & 1 & . & . & . \\ . & . & . & . & 1 & . & . \\ . & . & . & . & . & 1 & 1 \\ . & . & . & . & 1 & . & . \\ . & . & . & 1 & . & . & . \\ . & . & 1 & . & . & . & . \\ . & 1 & . & . & . & . & . \\ 1 & . & . & . & . & . & . \end{pmatrix}$$

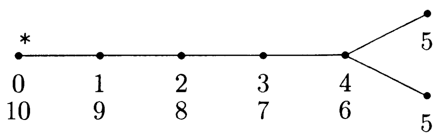
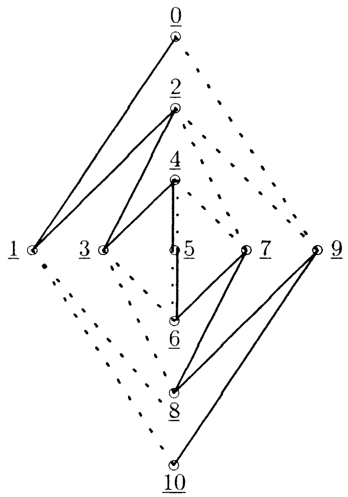


Fig. 19. Essential matrix E_0 and essential paths from the vertex 0 for the D_7 model.



$$W_{00} = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

Fig. 20. The D_7 Ocneanu graph and the modular invariant matrix.

It is spanned by a basis with 11 elements

$$\begin{aligned} &\underline{0} = 0\dot{\otimes}0, && \underline{4} = 4\dot{\otimes}0 = 0\dot{\otimes}4, && \underline{8} = 8\dot{\otimes}0 = 0\dot{\otimes}8, \\ &\underline{1} = 0\dot{\otimes}0 = 0\dot{\otimes}9, && \underline{5} = 5\dot{\otimes}0 = 0\dot{\otimes}5, && \underline{9} = 9\dot{\otimes}0 = 0\dot{\otimes}1, \\ &\underline{2} = 2\dot{\otimes}0 = 0\dot{\otimes}2, && \underline{6} = 6\dot{\otimes}0 = 0\dot{\otimes}6, && \underline{10} = 10\dot{\otimes}0 = 0\dot{\otimes}10. \\ &\underline{3} = 3\dot{\otimes}0 = 0\dot{\otimes}7, && \underline{7} = 7\dot{\otimes}0 = 0\dot{\otimes}3, && \end{aligned}$$

$1\dot{\otimes}0$ and $0\dot{\otimes}1$ are respectively the chiral left and right generators. The multiplication by these generators is encoded by the Ocneanu graph of D_7, represented in Fig. 20. All the points are ambichiral.

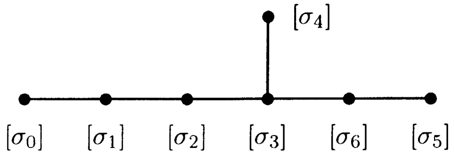
To obtain the toric matrices of the D_7 model, we need the essential matrices $E_i(A_{11})$ of the A_{11} case. We define new essential matrices $E_i^\rho(A_{11})$ defined by permuting the columns of $E_i(A_{11})$ associated to the vertices 1, 9, and 3, 7. The toric matrices of the D_7 model are then obtained by setting

$$W[a,b] \doteq E_a(A_{11}) \cdot (\widetilde{E_b^\rho(A_{11})}).$$

We recall the matrix expression of the modular invariant W_{00} and give the others as sesquilinears forms in the appendix.

8. The E_7 case

The E_7 Dynkin diagram and its adjacency matrix are displayed in Fig. 21. We use the following order for the vertices: $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_6, \sigma_5, \sigma_4\}$.



$$\mathcal{G}_{E_7} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Fig. 21. The E_7 Dynkin diagram and its adjacency matrix.

Here $\kappa = 18$, the norm of the graph is $\beta = [2]_q = 2\cos(\pi/18)$ and the normalized Perron–Frobenius vector is $D = ([1]_q, [2]_q, [3]_q, [4]_q, [6]_q/[2]_q, [4]_q/[3]_q, [4]_q/[2]_q)$.

The graph algebra of the Dynkin diagram E_7 is not a positive integral graph algebra. We give it for illustration but it will not be used in the sequel (Table 7).

The fusion matrices G_i are given by the following polynomials:

$$G_0 = Id_7, \qquad G_1 = \mathcal{G}, \qquad G_2 = G_1 \cdot G_1 - G_0, \qquad G_3 = G_1 \cdot G_2 - G_1,$$
$$G_5 = G_3 \cdot G_2 - G_1 - 2G_3, \qquad G_4 = G_5 \cdot G_2, \qquad G_6 = G_5 \cdot G_4 - G_2.$$

Essential matrices of E_7 have 7 columns and 17 rows. They are labelled by vertices of diagrams E_7 and A_{17}. The first essential matrix E_0 is given below, together with the corresponding induction–restriction graph. To obtain the toric matrices, we also need to know the essential matrices for the D_{10} case. They are obtained as usual (we also display the essential matrix E_0 of the D_{10} case) (Fig. 22).

We form the tensor product $D_{10} \otimes D_{10}$, and identify $au \otimes b$ with $a \otimes \rho(u)b$, where

$$\rho(0) = 0, \qquad \rho(2) = 8, \qquad \rho(4) = 4, \qquad \rho(6) = 6, \qquad \rho(8) = 2, \qquad \rho(8') = 8'.$$

The Ocneanu algebra of E_7 can be realized as

$$\mathcal{H}_{\mathrm{Oc}(E_7)} = D_{10} \dot{\otimes} D_{10} \doteq \frac{D_{10} \otimes D_{10}}{\rho}.$$

Table 7
Multiplication table of the graph algebra E_7

E_7	0	1	2	3	6	5	4
0	0	1	2	3	6	5	4
1	1	$0+2$	$1+3$	$2+4+6$	$3+5$	6	3
2	2	$1+3$	$0+2+4+6$	$1+3_2+5$	$2+4+6$	3	$2+6$
3	3	$2+4+6$	$1+3_2+5$	$0+2_2+4+6_2$	$1+3_2$	$2+4$	$1+3+5$
6	6	$3+5$	$2+4+6$	$1+3_2$	$0+2+6$	$1+5$	$2+4$
5	5	6	3	$2+4$	$1+5$	$0-4+6$	$3-5$
4	4	3	$2+6$	$1+3+5$	$2+4$	$3-5$	$0+6$

$$E_0(E_7) = \begin{pmatrix} 1 & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ . & . & 1 & . & . & . & . \\ . & . & . & 1 & . & . & . \\ . & . & . & . & 1 & . & 1 \\ . & . & . & 1 & . & 1 & . \\ . & . & 1 & . & 1 & . & . \\ . & 1 & . & 1 & . & . & . \\ 1 & . & 1 & . & . & . & 1 \\ . & 1 & . & 1 & . & . & . \\ . & . & 1 & . & 1 & . & . \\ . & . & . & 1 & . & 1 & . \\ . & . & . & . & 1 & . & 1 \\ . & . & . & 1 & . & . & . \\ . & . & 1 & . & . & . & . \\ . & 1 & . & . & . & . & . \\ 1 & . & . & . & . & . & . \end{pmatrix}$$

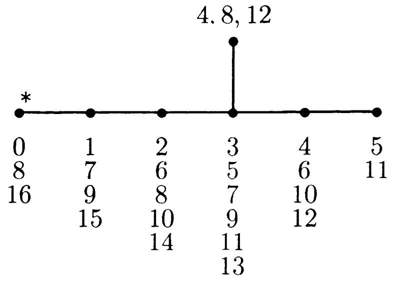


Fig. 22. Essential matrix E_0 and essential paths from the vertex 0 for the E_7-model.

It is spanned by a basis with 17 elements

$$\begin{array}{ll}
\underline{0} = 0\dot{\otimes}0, & \underline{(0)} = 0\dot{\otimes}1, \\
\underline{1} = 1\dot{\otimes}0, & \underline{(1)} = 1\dot{\otimes}1, \\
\underline{2} = 2\dot{\otimes}0 = 0\dot{\otimes}8, & \underline{(2)} = 2\dot{\otimes}1 = 0\dot{\otimes}7, \\
\underline{3} = 3\dot{\otimes}0, & \underline{(3)} = 3\dot{\otimes}1 = 1\dot{\otimes}3, \\
\underline{4} = 4\dot{\otimes}0 = 0\dot{\otimes}4, & \underline{(4)} = 0\dot{\otimes}3, \\
\underline{5} = 5\dot{\otimes}0, & \underline{(5)} = 5\dot{\otimes}1 - 3\dot{\otimes}1, \\
\underline{6} = 6\dot{\otimes}0 = 0\dot{\otimes}6, & \qquad = 1\dot{\otimes}5 - 1\dot{\otimes}3, \\
\underline{7} = 7\dot{\otimes}0, & \underline{(6)} = 0\dot{\otimes}5, \\
\underline{8} = 8\dot{\otimes}0 = 0\dot{\otimes}2, & \\
\underline{8'} = 8'\dot{\otimes}0 = 0\dot{\otimes}8'. &
\end{array}$$

$\underline{1}$ and $\underline{(0)}$ are respectively the left and right generators. The ambichiral part is the linear span of $\{\underline{0}, \underline{2}, \underline{4}, \underline{6}, \underline{8}, \underline{8'}\}$. The multiplication of the elements of this algebra by the generators is shown in the following table. We can observe on the Ocneanu graph $\mathrm{Oc}(E_7)$ that E_7 does not appear as a subalgebra of $\mathcal{H}_{\mathrm{Oc}(E_7)}$ but as a quotient (there are two such quotients) (Fig. 23).

The 17 toric matrices W_{ab} of the E_7 model are obtained as explained in Section 2.2.5, but with a twist. We use the essential matrices $E_a(D_{10})$, and replace the matrix elements of the columns associated with vertices 1, 3, 5, 7 of the graph D_{10} by 0; this being done, we permute the columns associated with vertices 2 and 8 of D_{10} (with our ordering, these are

$$E_0(D_{10}) = \begin{pmatrix} 1 & . & . & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . & . & . \\ . & . & . & 1 & . & . & . & . & . & . \\ . & . & . & . & 1 & . & . & . & . & . \\ . & . & . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & . & . & 1 & . & . \\ . & . & . & . & . & . & . & . & 1 & 1 \\ . & . & . & . & . & . & . & 1 & . & . \\ . & . & . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & 1 & . & . & . & . \\ . & . & . & . & 1 & . & . & . & . & . \\ . & . & . & 1 & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . & . & . \\ 1 & . & . & . & . & . & . & . & . & . \end{pmatrix}$$

Fig. 23. Essential matrix E_0 for D_{10}.

Table 8
Multiplication of the elements of the Ocneanu algebra of E_7 by the generators

"D_{10}"	$\underline{1}$
$\underline{0}$	$\underline{1}$
$\underline{1}$	$\underline{0}+\underline{2}$
$\underline{2}$	$\underline{1}+\underline{3}$
$\underline{3}$	$\underline{2}+\underline{4}$
$\underline{4}$	$\underline{3}+\underline{5}$
$\underline{5}$	$\underline{4}+\underline{6}$
$\underline{6}$	$\underline{5}+\underline{7}$
$\underline{7}$	$\underline{6}+\underline{8}+\underline{8'}$
$\underline{8}$	$\underline{7}$
$\underline{8'}$	$\underline{7}$
"E_7"	$\underline{1}$
$\underline{(0)}$	$\underline{(1)}$
$\underline{(1)}$	$\underline{(0)}+\underline{(2)}$
$\underline{(2)}$	$\underline{(1)}+\underline{(3)}$
$\underline{(3)}$	$\underline{(2)}+\underline{(4)}+\underline{(6)}$
$\underline{(4)}$	$\underline{(3)}$
$\underline{(6)}$	$\underline{(3)}+\underline{(5)}$
$\underline{(5)}$	$\underline{(6)}$

"D_{10}"	$\underline{(0)}$
$\underline{0}$	$\underline{(0)}$
$\underline{(0)}$	$\underline{0}+\underline{8}$
$\underline{8}$	$\underline{(0)}+\underline{(4)}$
$\underline{(4)}$	$\underline{8}+\underline{4}$
$\underline{4}$	$\underline{(4)}+\underline{(6)}$
$\underline{(6)}$	$\underline{4}+\underline{6}$
$\underline{6}$	$\underline{(6)}+\underline{(2)}$
$\underline{(2)}$	$\underline{2}+\underline{6}+\underline{8'}$
$\underline{2}$	$\underline{(2)}$
$\underline{8'}$	$\underline{(2)}$
"E_7"	$\underline{(0)}$
$\underline{1}$	$\underline{(1)}$
$\underline{(1)}$	$\underline{1}+\underline{7}$
$\underline{7}$	$\underline{(1)}+\underline{(3)}$
$\underline{(3)}$	$\underline{3}+\underline{5}+\underline{7}$
$\underline{3}$	$\underline{(3)}$
$\underline{5}$	$\underline{(3)}+\underline{(5)}$
$\underline{(5)}$	$\underline{5}$

columns 3 and 9). The reduced and twisted matrix so obtained is called $E_a^{\rho}(D_{10})$. The 17 toric matrices of the E_7 model are then obtained by setting:

$$W[a,b]\doteq E_a(D_{10})\cdot(\widetilde{E_b^{\rho}(D_{10})}).$$

For instance $W_{\underline{(5)}}=W[5,1]-W[3,1]=E_5(D_{10})\cdot(\widetilde{E_1^{\rho}(D_{10})})-E_3(D_{10})\cdot(\widetilde{E_1^{\rho}(D_{10})})$. We recall the matrix expression of the modular invariant W_{00} and give the others as sesquilinears forms in the appendix (Table 8).

$$W_{00}=\begin{pmatrix}
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1
\end{pmatrix}.$$

We conclude this section with the determination of the integers d_n and d_x.

The integers d_n associated with endomorphisms of essential paths on the graph E_7 are determined easily by the usual method. For $n=1,2,\ldots,17$ one finds

$$d_n=7,12,17,22,27,30,33,34,35,34,33,30,27,22,17,12,7.$$

In order to determine the integers d_x, we need to know the multiplication table defined by the Ocneanu graph $\mathrm{Oc}(E_7)$. The full table of $\mathcal{H}_{\mathrm{Oc}(E_7)}$ has the following structure: "D_{10}" × "D_{10}" → "D_{10}", "D_{10}" × "E_7" → "E_7", and "E_7" × "E_7" → "D_{10}", where "D_{10}" is the subalgebra linearly spanned by $\underline{0},\underline{1},\ldots,\underline{8},\underline{8'}$ and "E_7" is the linear subspace linearly spanned by $\underline{(0)},\underline{(1)},\ldots,\underline{(6)}$. Actually, it is enough, four our purpose, to know the smaller table obtained by restriction to the E_7 quotients, i.e., the "E_7" × "E_7" → "D_{10}" table.

We encode this multiplication table by a set of 10 matrices s_p (labelled by D_{10}, of size 7×7 (labelled by E_7)). The result of a given multiplication, such as $\underline{(4)}\times\underline{(5)}=\underline{3}+\underline{7}$ is indicated by the presence of the integer 1 in position (4, 5) in both matrices s_3 and s_7 (Fig. 24).

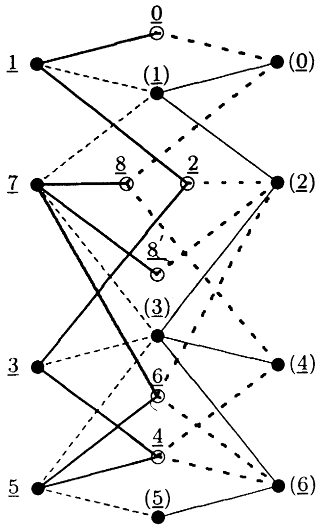


Fig. 24. Ocneanu graph E_7.

Since we have an explicit realization of $\mathcal{H}_{\mathrm{Oc}(E_7)}$, it is not too difficult to find

$$s_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad s_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix},$$

$$s_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}, \qquad s_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 0 \end{pmatrix},$$

$$s_4 = \begin{pmatrix} 0&0&0&0&1&0&1\\ 0&0&0&2&0&1&0\\ 0&0&2&0&1&0&2\\ 0&2&0&3&0&1&0\\ 1&0&1&0&1&0&1\\ 0&1&0&1&0&0&0\\ 1&0&2&0&1&0&1 \end{pmatrix}, \qquad s_5 = \begin{pmatrix} 0&0&0&1&0&1&0\\ 0&0&1&0&1&0&2\\ 0&1&0&3&0&1&0\\ 1&0&3&0&2&0&2\\ 0&1&0&2&0&0&0\\ 1&0&1&0&0&0&1\\ 0&2&0&2&0&1&0 \end{pmatrix},$$

$$s_6 = \begin{pmatrix} 0&0&1&0&0&0&1\\ 0&1&0&2&0&1&0\\ 1&0&2&0&2&0&2\\ 0&2&0&4&0&1&0\\ 0&0&2&0&1&0&1\\ 0&1&0&1&0&1&0\\ 1&0&2&0&1&0&2 \end{pmatrix}, \qquad s_7 = \begin{pmatrix} 0&1&0&1&0&0&0\\ 1&0&2&0&1&0&1\\ 0&2&0&3&0&1&0\\ 1&0&3&0&2&0&3\\ 0&1&0&2&0&1&0\\ 0&0&1&0&1&0&1\\ 0&1&0&3&0&1&0 \end{pmatrix},$$

$$s_8 = \begin{pmatrix} 1&0&0&0&1&0&0\\ 0&1&0&1&0&0&0\\ 0&0&2&0&0&0&1\\ 0&1&0&2&0&1&0\\ 1&0&0&0&1&0&1\\ 0&0&0&1&0&0&0\\ 0&0&1&0&1&0&1 \end{pmatrix}, \qquad s_{8'} = \begin{pmatrix} 0&0&1&0&0&0&0\\ 0&1&0&1&0&0&0\\ 1&0&1&0&1&0&1\\ 0&1&0&2&0&1&0\\ 0&0&1&0&0&0&1\\ 0&0&0&1&0&0&0\\ 0&0&1&0&1&0&1 \end{pmatrix}.$$

The sum of matrix elements of the 10 matrices s_p, for $p = 0, 1, 2, \ldots, 8, 8'$ is 7, 12, 17, 22, 27, 30, 33, 34, 18, 17, respectively.

To each linear generator $x = \sum a \otimes b$ (for instance $\underline{(5)} = 5\dot{\otimes}1 - 3\dot{\otimes}1$) of the Ocneanu algebra of E_7 (the basis with 17 elements was given previously), we associate a matrix $\Sigma_x = \sum s_a s_b$ (for instance $\Sigma_{\underline{(5)}} = s_5 s_1 - s_3 s_1$). The integer d_x is the sum of matrix elements of the matrix Σ_x (for instance $d_{\underline{(5)}} = 16$). In particular, the d_x associated with the "D_{10}" part of the graph are just given by sum of matrix elements of matrices s_p.

The final list of integers d_x, associated with blocks $\underline{0}, \underline{1}, \ldots, \underline{8}, \underline{8'}; \underline{(0)}, \underline{(1)}, \ldots, \underline{(6)}$ is

$$d_x = 7, 12, 17, 22, 27, 30, 33, 34, 18, 17, 12, 24, 34, 44, 22, 16, 30.$$

Note that $\sum d_n = \sum d_x = 399$ and that $\sum d_n^2 = \sum d_x^2 = 10905$.

The above results agree [6] with those obtained by [22].

[6] The preprint version of [22], available on the web, contains a typing misprint: the last values of d_x should be read 44, 30, 16, 22 and not 442, 30, 16, 22.

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Appendix A. The general notion of essential paths on a graph G

The general definitions given here are adapted from [16].

Call β the norm of the graph G (the biggest eigenvalue of the adjacency matrix $\mathcal{G}$) and D_i the components of the (normalized) Perron–Frobenius eigenvector. Call σ_i the vertices of G and, if σ_j is a neighbour of σ_i, call ξ_{ij} the oriented edge from σ_i to σ_j. If G is unoriented (the case for ADE and affine ADE diagrams), each edge should be considered as carrying both orientations.

An elementary path can be written either as a finite sequence of consecutive (neighbours on the graph) vertices, $[\sigma_{a_1}\sigma_{a_2}\sigma_{a_3}\dots]$, or as a sequence $(\xi(1)\xi(2)\dots)$ of consecutive edges, with $\xi(1) = \xi_{a_1a_2} = \sigma_{a_1}\sigma_{a_2}$, $\xi(2) = \xi_{a_2a_3} = \sigma_{a_2}\sigma_{a_3}$, etc. Vertices are considered as paths of length 0.

The length of the (possibly backtracking) path $(\xi(1)\xi(2)\dots\xi(p))$ is p. We call $r(\xi_{ij}) = \sigma_j$, the range of ξ_{ij} and $s(\xi_{ij}) = \sigma_i$, the source of ξ_{ij}.

For all edges $\xi(n+1) = \xi_{ij}$ that appear in an elementary path, we set $\xi(n+1)^{-1} \dot{=} \xi ji$.

For every integer $n > 0$, the annihilation operator C_n, acting on elementary paths of length p is defined as follows: if $p \le n$, C_n vanishes and if $p \ge n+1$ then

$$\begin{aligned} &C_n(\xi(1)\xi(2)\dots\xi(n)\xi(n+1)\dots) \\ &\quad = \sqrt{\frac{D_{r(\xi(n))}}{D_{s(\xi(n))}}}\delta_{\xi(n),\xi(n+1)^{-1}}(\xi(1)\xi(2)\dots\hat{\xi}(n)\hat{\xi}(n+1)\dots). \end{aligned}$$

Here, the symbol "hat" (like in $\hat{\xi}$) denotes omission. The result is therefore either 0 or a linear combination of paths of length $p-2$. Intuitively, C_n chops the round trip that possibly appears at positions $n, n+1$.

A path is called essential if it belongs to the intersection of the kernels of the annihilators C_n's.

For instance, in the case of the diagram E_6,

$$C_3(\xi_{01}\xi_{12}\xi_{23}\xi_{32}) = \sqrt{\frac{1}{[2]}}(\xi_{01}\xi_{12}), \qquad C_3(\xi_{01}\xi_{12}\xi_{25}\xi_{52}) = \sqrt{\frac{[2]}{[3]}}(\xi_{01}\xi_{12}).$$

The following difference of non-essential paths of length 4 starting at σ_0 and ending at σ_2 is an essential path of length 4 on E_6:

$$\sqrt{[2]}(\xi_{01}\xi_{12}\xi_{23}\xi_{32}) - \sqrt{\frac{[3]}{[2]}}(\xi_{01}\xi_{12}\xi_{25}\xi_{52}) = \sqrt{[2]}[0,1,2,3,2] - \sqrt{\frac{[3]}{[2]}}[0,1,2,5,2].$$

Here the values of q-numbers are $[2] = \sqrt{2}/(\sqrt{3}-1)$ and $[3] = 2/(\sqrt{3}-1)$.

Acting on an elementary path of length p, the creating operators $C_n^\dagger$ are defined as follows: if $n > p+1$, $C_n^\dagger$ vanishes and, if $n \leq p+1$ then, setting $j = r(\xi(n-1))$,

$$C_n^\dagger(\xi(1)\ldots\xi(n-1)\ldots) = \sum_{d(j,k)=1} \sqrt{\left(\frac{D_k}{D_j}\right)}(\xi(1)\ldots\xi(n-1)\xi_{jk}\xi_{kj}\ldots).$$

The above sum is taken over the neighbours σ_k of σ_j on the graph. Intuitively, this operator adds one (or several) small round trip(s) at position n. The result is therefore either 0 or a linear combination of paths of length $p+2$.

For instance, on paths of length 0 (i.e., vertices),

$$C_1^\dagger(\sigma_j) = \sum_{d(j,k)=1} \sqrt{\left(\frac{D_k}{D_j}\right)}\xi_{jk}\xi_{kj} = \sum_{d(j,k)=1} \sqrt{\left(\frac{D_k}{D_j}\right)}[\sigma_j\sigma_k\sigma_j].$$

Jones' projectors e_k can be defined (as endomorphisms of Path^p) by

$$e_k \doteq \frac{1}{\beta} C_k^\dagger C_k.$$

The reader can check that all Jones–Temperley–Lieb relations between the e_i are satisfied. Essential paths can also be defined as elements of the intersection of the kernels of the Jones projectors e_i's.

Appendix B. Twisted partition functions for the ADE models

Twisted partition functions for the A_4 model

Point	$\mathcal{Z}$
$\underline{0}$	$\lvert\chi_0\rvert^2+\lvert\chi_1\rvert^2+\lvert\chi_2\rvert^2+\lvert\chi_3\rvert^2$
$\underline{1}$	$[(\chi_0\overline{\chi_1}+\chi_1\overline{\chi_2}+\chi_2\overline{\chi_3})+\text{h.c.}]$
$\underline{2}$	$\lvert\chi_1\rvert^2+\lvert\chi_2\rvert^2+[(\chi_0\overline{\chi_2}+\chi_1\overline{\chi_3})+\text{h.c.}]$
$\underline{3}$	$[(\chi_0\overline{\chi_3}+\chi_1\overline{\chi_2})+\text{h.c.}]$

Twisted partition functions for the E_6 model

Point	$\mathcal{Z}$
$\underline{0}$	$\lvert\chi_0+\chi_6\rvert^2+\lvert\chi_3+\chi_7\rvert^2+\lvert\chi_4+\chi_{10}\rvert^2$
$\underline{3}$	$(\chi_0+\chi_4+\chi_6+\chi_{10})\cdot(\overline{\chi_3}+\overline{\chi_7})+\text{h.c.}$
$\underline{4}$	$\lvert\chi_3+\chi_7\rvert^2+[(\chi_0+\chi_6)\cdot(\overline{\chi_4}+\overline{\chi_{10}})+\text{h.c.}]$
$\underline{11'}$	$\lvert\chi_1+\chi_5+\chi_7\rvert^2+\lvert\chi_2+\chi_4+\chi_6+\chi_8\rvert^2+\lvert\chi_3+\chi_5+\chi_9\rvert^2$
$\underline{21'}$	$(\chi_1+\chi_3+2(\chi_5)+\chi_7+\chi_9)\cdot(\overline{\chi_2}+\overline{\chi_4}+\overline{\chi_6}+\overline{\chi_8})+\text{h.c.}$
$\underline{51'}$	$\lvert\chi_2+\chi_4+\chi_6+\chi_8\rvert^2+2\lvert\chi_5\rvert^2+([(\chi_1+\chi_7)\cdot(\overline{\chi_3}+\overline{\chi_5}+\overline{\chi_9})+\chi_3\overline{\chi_5}+\chi_5\overline{\chi_9}]$ $+\text{h.c.})$

Appendix B (*Continued*)

Point	$\mathcal{Z}$
$\underline{1}$	$(\chi_0+\chi_6)\cdot(\overline{\chi_1}+\overline{\chi_5}+\overline{\chi_7})+(\chi_3+\chi_7)\cdot(\overline{\chi_2}+\overline{\chi_4}+\overline{\chi_6}+\overline{\chi_8})+(\chi_4+\chi_{10})\cdot(\overline{\chi_3}+\overline{\chi_5}+\overline{\chi_9})$
$\underline{1'}$	h.c.$(Z_{\underline{1}})$
$\underline{2}$	$\lvert\chi_3+\chi_7\rvert^2+\lvert\chi_4+\chi_6\rvert^2+(\chi_0+\chi_{10})\cdot(\overline{\chi_2}+\overline{\chi_4}+\overline{\chi_6}+\overline{\chi_8})+(\chi_3+\chi_7)\cdot(\overline{\chi_1}+2(\overline{\chi_5})+\overline{\chi_9})+(\chi_4+\chi_6)\cdot(\overline{\chi_2}+\overline{\chi_8})$
$\underline{31'}$	h.c.$(Z_{\underline{2}})$
$\underline{5}$	$(\chi_0+\chi_6)\cdot(\overline{\chi_3}+\overline{\chi_5}+\overline{\chi_9})+(\chi_3+\chi_7)\cdot(\overline{\chi_2}+\overline{\chi_4}+\overline{\chi_6}+\overline{\chi_8})+(\chi_4+\chi_{10})\cdot(\overline{\chi_1}+\overline{\chi_5}+\overline{\chi_7})$
$\underline{41'}$	h.c.$(Z_{\underline{5}})$

Twisted partition functions for the E_8 model (part 1)

Point	$\mathcal{Z}$
$\underline{0}$	$\lvert\chi_0+\chi_{10}+\chi_{18}+\chi_{28}\rvert^2+\lvert\chi_6\rvert+\chi_{12}+\chi_{16}+\chi_{22}\rvert^2$
$\underline{6}$	$\lvert\chi_0+\chi_6+\chi_{12}+\chi_{16}+\chi_{22}+\chi_{28}\rvert^2-\lvert\chi_0+\chi_{28}\rvert^2+[(\chi_6+\chi_{12}+\chi_{16}+\chi_{22})\cdot(\overline{\chi_{10}}+\overline{\chi_{18}})+\text{h.c.}]$
$\underline{11'}$	$\lvert\chi_1+\chi_9+\chi_{11}+\chi_{17}+\chi_{19}+\chi_{27}\rvert^2+\lvert\chi_5+\chi_7+\chi_{11}+\chi_{13}+\chi_{15}+\chi_{17}+\chi_{21}+\chi_{23}$
$\underline{71'}$	$\lvert\chi_1+\chi_5+\chi_7+\chi_{11}+\chi_{13}+\chi_{15}+\chi_{17}+\chi_{21}+\chi_{23}+\chi_{27}\rvert^2+\lvert\chi_9+\chi_{11}+\chi_{13}+\chi_{15}+\chi_{17}+\chi_{19}\rvert^2-\lvert\chi_1+\chi_{27}\rvert^2+\lvert\chi_{11}+\chi_{17}\rvert^2-\lvert\chi_{13}+\chi_{15}\rvert^2-\lvert\chi_9+\chi_{19}\rvert^2+[(\chi_5+\chi_7+\chi_{21}+\chi_{23})\cdot(\overline{\chi_9}+\overline{\chi_{11}}+\overline{\chi_{17}}+\overline{\chi_{19}})+\text{h.c.}]$
$\underline{22'}$	$\lvert\chi_2+\chi_{26}\rvert^2+\lvert\sum_{i=2}^{12}(\chi_{2i})\rvert^2+\lvert\sum_{i=4}^{10}(\chi_{2i})\rvert^2+2\lvert\chi_{14}\rvert^2+[((\chi_2+\chi_{26})\cdot(\overline{\chi_8}+\overline{\chi_{10}}+\overline{\chi_{12}}+\overline{\chi_{16}}+\overline{\chi_{18}}+\overline{\chi_{20}})+(\chi_4+\chi_6+\chi_{22}+\chi_{24})\cdot(\overline{\chi_{14}}))+\text{h.c.}]$
$\underline{42'}$	$\lvert\sum_{i=1}^{13}(\chi_{2i})\rvert^2-\lvert\chi_2+\chi_{24}\rvert^2+\lvert\sum_{i=2}^{12}(\chi_{2i})\rvert^2-\lvert\chi_4+\chi_6+\chi_{22}+\chi_{24}\rvert^2+\lvert\sum_{i=4}^{10}(\chi_{2i})\rvert^2+[(\chi_2+\sum_{i=4}^{10}(\chi_{2i})+\chi_{26})\cdot\overline{\chi_{14}}+\text{h.c.}]$
$\underline{55'}$	$\lvert\chi_5+\chi_9+\chi_{13}+\chi_{15}+\chi_{19}+\chi_{23}+\lvert\chi_3+\sum_{i=3}^{10}(\chi_{2i+1})+\chi_{25}\rvert^2$
$\underline{35'}$	$\lvert\sum_{i=1}^{12}(\chi_{2i+1})\rvert^2+\lvert\sum_{i=3}^{10}(\chi_{2i+1})\rvert^2+\lvert\chi_9+\chi_{13}+\chi_{15}+\chi_{19}\rvert^2-\lvert\chi_7+\chi_{11}+\chi_{17}+\chi_{21}\rvert^2-\lvert\chi_5+\chi_{23}\rvert+[(\chi_3+\chi_{25})\cdot(\overline{\chi_9}+\overline{\chi_{13}}+\overline{\chi_{15}}+\overline{\chi_{19}})+\text{h.c.}]$
$\underline{1}$	$(\chi_0+\chi_{10}+\chi_{18}+\chi_{28})\cdot(\overline{\chi_1}+\overline{\chi_9}+\overline{\chi_{11}}+\overline{\chi_{17}}+\overline{\chi_{19}}+\overline{\chi_{27}})+(\chi_6+\chi_{12}+\chi_{16}+\chi_{22})\cdot(\overline{\chi_5}+\overline{\chi_7}+\overline{\chi_{11}}+\overline{\chi_{13}}+\overline{\chi_{15}}+\overline{\chi_{17}}+\overline{\chi_{21}}+\overline{\chi_{23}})$
$\underline{1'}$	h.c.$(Z_{\underline{1}})$
$\underline{2}$	$\lvert\chi_6+\chi_{12}+\chi_{16}+\chi_{22}\rvert^2+(\chi_0+\chi_{10}+\chi_{18}+\chi_{28})\cdot(\overline{\chi_2}+\overline{\chi_8}+\overline{\chi_{10}}+\overline{\chi_{12}}+\overline{\chi_{16}}+\overline{\chi_{18}}+\overline{\chi_{20}}+\overline{\chi_{26}})+(\chi_6+\chi_{12}+\chi_{16}+\chi_{22})\cdot(\overline{\chi_4}+\overline{\chi_8}+\overline{\chi_{10}}+2(\overline{\chi_{14}})+\overline{\chi_{18}}+\overline{\chi_{20}}+\overline{\chi_{24}})$
$\underline{2'}$	h.c.$(Z_{\underline{2}})$
$\underline{3}$	$(\chi_0+\chi_{10}+\chi_{18}+\chi_{28})\cdot(\overline{\chi_3}+\sum_{i=3}^{10}(\overline{\chi_{2i+1}})+\overline{\chi_{25}})+(\chi_6+\chi_{12}+\chi_{16}+\chi_{22})\cdot(\sum_{i=1}^{12}(\overline{\chi_{2i+1}})+\overline{\chi_9}+\overline{\chi_{13}}+\overline{\chi_{15}}+\overline{\chi_{19}})$
$\underline{65'}$	h.c.$(Z_{\underline{3}})$
$\underline{4}$	$\lvert\chi_6+\chi_{10}+\chi_{12}+\chi_{16}+\chi_{18}+\chi_{22}\rvert^2+\lvert\chi_{12}+\chi_{16}\rvert^2+(\chi_6+\chi_{22})\cdot(\overline{\chi_{12}}+\overline{\chi_{16}})+(\chi_0+\chi_{10}+\chi_{18}+\chi_{28})\cdot(\overline{\chi_4}+\overline{\chi_8}+2(\overline{\chi_{14}})+\overline{\chi_{20}}+\overline{\chi_{24}})+(\chi_0+\chi_{28})\cdot(\overline{\chi_6}+\overline{\chi_{10}}+\overline{\chi_{12}}+\overline{\chi_{16}}+\overline{\chi_{18}}+\overline{\chi_{22}})+(\chi_6+\chi_{12}+\chi_{16}+\chi_{22})\cdot(\overline{\chi_2}+\overline{\chi_4}+2(\overline{\chi_8})+\overline{\chi_{10}}+2(\overline{\chi_{14}})+\overline{\chi_{18}}+2(\overline{\chi_{20}})+\overline{\chi_{24}}+\overline{\chi_{26}})$
$\underline{62'}$	h.c.$(Z_{\underline{4}})$

Twisted partition functions for the E_8 model (part 2)

Point	Z
$\underline{5}$	$(\chi_0+\chi_{10}+\chi_{18}+\chi_{28})\cdot(\overline{\chi_5}+\overline{\chi_9}+\overline{\chi_{13}}+\overline{\chi_{15}}+\overline{\chi_{19}}+\overline{\chi_{23}})+(\chi_6+\chi_{12}+\chi_{16}+\chi_{22})\cdot(\overline{\chi_3}+\sum_{i=3}^{10}(\overline{\chi_{2i+1}})+\overline{\chi_{25}})$
$\underline{5'}$	h.c.$(Z_{\underline{5}})$
$\underline{7}$	$(\chi_0+\chi_{10}+\chi_{18}+\chi_{28})\cdot(\overline{\chi_5}+\overline{\chi_7}+\overline{\chi_{11}}+\overline{\chi_{13}}+\overline{\chi_{15}}+\overline{\chi_{17}}+\overline{\chi_{21}}+\overline{\chi_{23}})+(\chi_6+\chi_{12}+\chi_{16}+\chi_{22})\cdot(\overline{\chi_1}+\sum_{i=2}^{11}(\overline{\chi_{2i+1}})+\overline{\chi_{11}}+\overline{\chi_{17}}+\overline{\chi_{27}})$
$\underline{61'}$	h.c.$(Z_{\underline{7}})$
$\underline{21'}$	$(\chi_1+\chi_9+\chi_{11}+\chi_{17}+\chi_{19}+\chi_{27})\cdot(\overline{\chi_2}+\overline{\chi_8}+\overline{\chi_{10}}+\overline{\chi_{12}}+\overline{\chi_{16}}+\overline{\chi_{18}}+\overline{\chi_{20}}+\overline{\chi_{26}})+(\chi_5+\chi_7+\chi_{11}+\chi_{13}+\chi_{15}+\chi_{17}+\chi_{21}+\chi_{23})\cdot(\sum_{i=2}^{12}(\overline{\chi_{2i}})+\overline{\chi_{14}})$
$\underline{12'}$	h.c.$(Z_{\underline{21'}})$
$\underline{41'}$	$(\chi_1+\chi_9+\chi_{11}+\chi_{17}+\chi_{19}+\chi_{27})\cdot(\sum_{i=2}^{12}(\overline{\chi_{2i}})+\overline{\chi_{14}})+(\chi_5+\chi_7+\chi_{11}+\chi_{13}+\chi_{15}+\chi_{17}+\chi_{21}+\chi_{23})\cdot(\sum_{i=1}^{13}(\overline{\chi_{2i}})+\sum_{i=4}^{10}(\overline{\chi_{2i}}))$
$\underline{72'}$	h.c.$(Z_{\underline{41'}})$
$\underline{52'}$	$(\chi_2+\chi_8+\chi_{10}+\chi_{12}+\chi_{16}+\chi_{18}+\chi_{20}+\chi_{26})\cdot(\overline{\chi_5}+\overline{\chi_9}+\overline{\chi_{13}}+\overline{\chi_{15}}+\overline{\chi_{19}}+\overline{\chi_{23}})+(\sum_{i=2}^{12}(\chi_{2i})+\chi_{14})\cdot(\overline{\chi_3}+\sum_{i=3}^{10}(\overline{\chi_{2i+1}})+\overline{\chi_{25}})$
$\underline{25'}$	h.c.$(Z_{\underline{52'}})$
$\underline{32'}$	$(\chi_2+\chi_8+\chi_{10}+\chi_{12}+\chi_{16}+\chi_{18}+\chi_{20}+\chi_{26})\cdot(\overline{\chi_3}+\sum_{i=3}^{10}(\overline{\chi_{2i+1}})+\overline{\chi_{25}})+(\sum_{i=2}^{12}(\chi_{2i})+\chi_{14})\cdot(\sum_{i=1}^{12}(\overline{\chi_{2i+1}})+\overline{\chi_9}+\overline{\chi_{13}}+\overline{\chi_{15}}+\overline{\chi_{19}})$
$\underline{45'}$	h.c.$(Z_{\underline{32'}})$
$\underline{51'}$	$(\chi_1+\chi_9+\chi_{11}+\chi_{17}+\chi_{19}+\chi_{27})\cdot(\overline{\chi_5}+\overline{\chi_9}+\overline{\chi_{13}}+\overline{\chi_{15}}+\overline{\chi_{19}}+\overline{\chi_{23}})+(\chi_5+\chi_7+\chi_{11}+\chi_{13}+\chi_{15}+\chi_{17}+\chi_{21}+\chi_{23})\cdot(\overline{\chi_3}+\sum_{i=3}^{10}(\overline{\chi_{2i+1}})+\overline{\chi_{25}})$
$\underline{15'}$	h.c.$(Z_{\underline{51'}})$
$\underline{31'}$	$\lvert\sum_{i=2}^{11}(\chi_{2i+1})\rvert^2+(\chi_1+\chi_{11}+\chi_{17}+\chi_{27})\cdot(\overline{\chi_3}+\sum_{i=3}^{10}(\overline{\chi_{2i+1}})+\overline{\chi_{25}})+(\chi_9+\chi_{19})\cdot(\overline{\chi_3}-\overline{\chi_5}-\overline{\chi_{23}}+\overline{\chi_{25}})(\chi_5+\chi_7+\chi_{11}+\chi_{13}+\chi_{15}+\chi_{17}+\chi_{21}+\chi_{23})\cdot(\overline{\chi_3}+\overline{\chi_9}+\overline{\chi_{13}}+\overline{\chi_{15}}+\overline{\chi_{19}}+\overline{\chi_{25}})$
$\underline{75'}$	h.c.$(Z_{\underline{31'}})$

Twisted partition functions for the D_4 model

Point	Z
$\underline{0}$	$\lvert\chi_0+\chi_4+2\lvert\chi_2\rvert^2$
$\underline{2}, \underline{2'}$	$\lvert\chi_2\rvert^2+[(\chi_0+\chi_4)\cdot\overline{\chi_2}+\text{h.c.}]$
$\underline{\epsilon}, \underline{2\epsilon}, \underline{2'\epsilon}$	$\lvert\chi_1+\chi_3\rvert^2$
$\underline{1}$	$(\chi_0+2(\chi_2)+\chi_4)\cdot(\overline{\chi_1}+\overline{\chi_3})$
$\underline{1\epsilon}$	h.c.$(Z_{\underline{1}})$

Appendix B (*Continued*)

Twisted partition functions for the D_6 model

Point	Z
$\underline{0}$	$\lvert\chi_0+\chi_8\rvert^2+\lvert\chi_2+\chi_6\rvert^2+2\lvert\chi_4\rvert^2$
$\underline{2}$	$\lvert\chi_2+\chi_6\rvert^2+2\lvert\chi_4\rvert^2+[(\chi_0+2(\chi_4)+\chi_8)\cdot(\overline{\chi_2}+\overline{\chi_6})+\text{h.c.}]$
$\underline{4}$, $\underline{4'}$	$\lvert\chi_2+\chi_4+\chi_6\rvert^2+[(\chi_0+\chi_8)\cdot\overline{\chi_4})+\text{h.c.}]$
$\underline{\epsilon}$	$\lvert\chi_1+\chi_7\rvert^2+\lvert\chi_3+\chi_5\rvert^2$
$\underline{2\epsilon}$	$\lvert\chi_1+\chi_3+\chi_5+\chi_7\rvert^2+\lvert\chi_3+\chi_5\rvert^2$
$\underline{4\epsilon}$, $\underline{4'\epsilon}$	$\lvert\chi_1+\chi_3+\chi_5+\chi_7\rvert^2-\lvert\chi_1+\chi_7\rvert^2$
$\underline{1}$	$(\chi_0+\chi_2+\chi_6+\chi_8)\cdot(\overline{\chi_1}+\overline{\chi_7})+(\chi_2+2(\chi_4)+\chi_6)\cdot(\overline{\chi_3}+\overline{\chi_5})$
$\underline{1\epsilon}$	$\text{h.c.}(Z_{\underline{1}})$
$\underline{3}$	$(\chi_0+\chi_2+\chi_6+\chi_8)\cdot(\overline{\chi_3}+\overline{\chi_5})+(\chi_2+2(\chi_4)+\chi_6)\cdot(\overline{\chi_1}+\overline{\chi_3}+\overline{\chi_5}+\overline{\chi_7})$
$\underline{3\epsilon}$	$\text{h.c.}(Z_{\underline{3}})$

Twisted partition functions for the D_5 model

Point	$\mathcal{Z}$
$\underline{0}$	$\lvert\chi_0\rvert^2+\lvert\chi_2\rvert^2+\lvert\chi_3\rvert^2+\lvert\chi_4\rvert^2+\lvert\chi_6\rvert^2+(\chi_1\overline{\chi_5}+\text{h.c.})$
$\underline{2}$	$\lvert\chi_2+\chi_4\rvert^2+\lvert\chi_3\rvert^2+[(\chi_0\overline{\chi_2}+\chi_1\overline{\chi_3}+\chi_1\overline{\chi_5}+\chi_3\overline{\chi_5}+\chi_4\overline{\chi_6})+\text{h.c.}]$
$\underline{3}$	$[(\chi_0+\chi_2+\chi_4+\chi_6)\cdot\overline{\chi_3}+(\chi_1+\chi_5)\cdot(\overline{\chi_2}+\chi_4)+\text{h.c.}]$
$\underline{4}$	$\lvert\chi_1\rvert^2+\lvert\chi_3\rvert^2+\lvert\chi_5\rvert^2+\lvert\chi_2+\chi_4\rvert^2+[(\chi_0\overline{\chi_4}+\chi_1\overline{\chi_3}+\chi_2\overline{\chi_6}+\chi_3\overline{\chi_5})+\text{h.c.}]$
$\underline{6}$	$\lvert\chi_1\rvert^2+\lvert\chi_3\rvert^2+\lvert\chi_5\rvert^2+[(\chi_0\overline{\chi_6}+\chi_2\overline{\chi_4})+\text{h.c.}]$
$\underline{1}$	$(\chi_0+\chi_2)\cdot\overline{\chi_1}+\chi_5\cdot(\overline{\chi_0}+\overline{\chi_2})+\chi_1\cdot(\overline{\chi_4}+\overline{\chi_6})+(\chi_4+\chi_6)\cdot\overline{\chi_5}$ $+[(\chi_2\overline{\chi_3}+\chi_3\overline{\chi_4})+\text{h.c.}]$
$\underline{5}$	$\text{h.c.}(Z_{\underline{1}})$

Twisted partition functions for the D_7 model

Point	$\mathcal{Z}$
$\underline{0}$	$\lvert\chi_0\rvert^2+\lvert\chi_2\rvert^2+\lvert\chi_4\rvert^2+\lvert\chi_5\rvert^2+\lvert\chi_6\rvert^2+\lvert\chi_8\rvert^2+\lvert\chi_{10}\rvert^2+[(\chi_1\overline{\chi_9}+\chi_3\overline{\chi_7})+\text{h.c.}]$
$\underline{2}$	$\lvert\chi_2+\chi_4\rvert^2+\lvert\chi_5\rvert^2+\lvert\chi_6+\chi_8\rvert^2+[(\chi_0\overline{\chi_2}+(\chi_1+\chi_3)\cdot(\overline{\chi_7}+\overline{\chi_9})+\chi_3\overline{\chi_5}+\chi_4\overline{\chi_6}$ $+\chi_5\overline{\chi_7}+\chi_8\overline{10})+\text{h.c.}]$
$\underline{4}$	$\lvert\chi_2+\chi_4+\chi_6+\chi_8\rvert^2+\lvert\chi_3+\chi_5+\chi_7\rvert^2+[\chi_0\overline{\chi_4}+\chi_1\cdot(\overline{\chi_5}+\overline{\chi_7})-\chi_2\overline{\chi_8}$ $+(\chi_3+\chi_5)\cdot\overline{\chi_9}+\chi_6\overline{\chi_{10}})+\text{h.c.}]$
$\underline{5}$	$[(\chi_0+\chi_2+\chi_4+\chi_6+\chi_8+\chi_{10})\cdot\overline{\chi_5}+(\chi_1+\chi_3+\chi_7+\chi_9)\cdot(\overline{\chi_4}+\overline{\chi_6})$ $+(\chi_2+\chi_8)\cdot(\overline{\chi_3}+\overline{\chi_7})+\text{h.c.}]$
$\underline{6}$	$\lvert\chi_2+\chi_4+\chi_6+\chi_8\rvert^2+\lvert\chi_3+\chi_5+\chi_7\rvert^2-\lvert\chi_2\rvert^2-\lvert\chi_8\rvert^2+[(\chi_0\overline{\chi_6}+\chi_1$ $\cdot(\overline{\chi_3}+\overline{\chi_5})+\chi_4\overline{\chi_{10}}+(\chi_5+\chi_7)\cdot\overline{\chi_9})+\text{h.c.}]$
$\underline{8}$	$\lvert\chi_1+\chi_3\rvert^2+\lvert\chi_4+\chi_6\rvert^2+\chi_7+\chi_9\rvert^2+\lvert\chi_5\rvert^2+[(\chi_0\overline{\chi_8}+\chi_2\cdot(\overline{\chi_6}+\overline{\chi_8}+\overline{\chi_{10}})$ $+\chi_3\overline{\chi_5}+\chi_4\overline{\chi_8}+\chi_5\overline{\chi_7})+\text{h.c.}]$
$\underline{10}$	$\lvert\chi_1\rvert^2+\lvert\chi_3\rvert^2+\lvert\chi_5\rvert^2+\lvert\chi_7\rvert^2+\lvert\chi_9\rvert^2+[(\chi_0\overline{\chi_{10}}+\chi_2\overline{\chi_8}+\chi_4\overline{\chi_6})+\text{h.c.}]$

Point	$\mathcal{Z}$
$\underline{1}$	$(\chi_0+\chi_2)\cdot\overline{\chi_1}+\chi_9\cdot(\overline{\chi_0}+\overline{\chi_2})+(\chi_2+\chi_4)\cdot\overline{\chi_3}+\chi_7\cdot(\overline{\chi_2}+\overline{\chi_4})+(\chi_6+\chi_8)\cdot\overline{\chi_7}+\chi_3\cdot(\overline{\chi_6}+\overline{\chi_8})+(\chi_8+\chi_{10})\cdot\overline{\chi_9}+\chi_1\cdot(\overline{\chi_8}+\overline{\chi_{10}})+[(\chi_4+\chi_6)\cdot\overline{\chi_5}+\text{h.c.}]$
$\underline{9}$	$\text{h.c.}(Z_{\underline{1}})$
$\underline{3}$	$\chi_7\cdot(\overline{\chi_0}+\overline{\chi_2})+(\chi_2+\chi_4)\cdot\overline{\chi_1}+\chi_9\cdot(\overline{\chi_2}+\overline{\chi_4})+(\chi_0+\chi_2)\cdot\overline{\chi_3}+\chi_1\cdot(\overline{\chi_6}+\overline{\chi_8})+(\chi_8+\chi_{10})\cdot\overline{\chi_7}+\chi_3\cdot(\overline{\chi_8}+\overline{\chi_{10}})+(\chi_6+\chi_8)\cdot\overline{\chi_9}+[(\chi_2+\chi_4+\chi_6+\chi_8)\cdot\overline{\chi_5}+(\chi_3+\chi_7)\cdot(\chi_4+\chi_6)+\text{h.c.}]$
$\underline{7}$	$\text{h.c.}(Z_{\underline{3}})$

Twisted partition functions for the E_7 model

Point	$\mathcal{Z}$
$\underline{0}$	$\lvert\chi_0+\chi_{16}\rvert^2+\lvert\chi_4+\chi_{12}\rvert^2+\lvert\chi_6+\chi_{10}\rvert^2+\lvert\chi_8\rvert^2+[(\chi_2+\chi_{14})\cdot\overline{\chi_8}+\text{h.c.}]$
$\underline{2}$	$\lvert\chi_4+\chi_6+\chi_8+\chi_{10}+\chi_{12}-(\chi_4+\chi_{12})\cdot\overline{\chi_8}+\chi_8\cdot(\overline{\chi_0}+\overline{\chi_{16}})+(\chi_0+\chi_4+\chi_8+\chi_{12}+\chi_{16})\cdot(\overline{\chi_2}+\overline{\chi_{14}})+(\chi_2+\chi_{14})\cdot(\overline{\chi_6}+\overline{\chi_8}+\overline{\chi_{10}})+(\chi_6+\chi_{10})\cdot\overline{\chi_8}$
$\underline{4}$	$\lvert\chi_2+\chi_4+\chi_6+\chi_8+\chi_{10}+\chi_{12}+\chi_{14}\rvert^2-\lvert\chi_2+\chi_{14}\rvert^2+\lvert\chi_6+\chi_8+\chi_{10}\rvert^2-\lvert\chi_8\rvert^2+[(\chi_0+\chi_8+\chi_{16})\cdot(\overline{\chi_4}+\overline{\chi_{12}})+\text{h.c.}]$
$\underline{6}$	$\lvert\chi_2+\chi_4+\chi_6+\chi_8+\chi_{10}+\chi_{12}+\chi_{14}\rvert^2+\lvert\chi_4+\chi_6+\chi_8+\chi_{10}+\chi_{12}\rvert^2-\lvert\chi_4+\chi_{12}\rvert^2+\lvert\chi_8\rvert^2+[(\chi_0+\chi_{16})\cdot(\overline{\chi_6}+\overline{\chi_{12}})+\text{h.c.}]$
$\underline{8}$	$\lvert\chi_4+\chi_6+\chi_8+\chi_{10}+\chi_{12}\rvert^2-\chi_8\cdot(\overline{\chi_4}+\overline{\chi_{12}})+(\chi_2+\chi_{14})\cdot(\overline{\chi_0}+\overline{\chi_4}+\overline{\chi_8}+\overline{\chi_{12}}+\overline{\chi_{16}})+(\chi_6+\chi_8+\chi_{10})\cdot(\overline{\chi_2}+\overline{\chi_{14}})+(\chi_0+\chi_{16})\cdot\overline{\chi_8}+\chi_8\cdot(\overline{\chi_6}+\overline{\chi_{10}})$
$\underline{8'}$	$\lvert\chi_2+\chi_6+\chi_{10}+\chi_{14}\rvert^2-\lvert\chi_6+\chi_{10}\rvert^2+\lvert\chi_4+\chi_6+\chi_8+\chi_{10}+\chi_{12}\rvert^2+\lvert\chi_8\rvert^2+[(\chi_0+\chi_{16})\cdot\overline{\chi_8}+\text{h.c.}]$
$(\underline{1})$	$\lvert\chi_1+\chi_7+\chi_9+\chi_{15}\rvert^2+\lvert\chi_3+\chi_5+\chi_7+\chi_9+\chi_{11}+\chi_{13}\rvert^2+\lvert\chi_5+\chi_{11}\rvert^2$
$(\underline{3})$	$\lvert\sum_{i=0}^{7}(\chi_{2i+1})\rvert^2+\lvert\sum_{i=1}^{6}(\chi_{2i+1})\rvert^2+\lvert\sum_{i=2}^{5}(\chi_{2i+1})\rvert^2-\lvert\chi_1+\chi_{15}\rvert^2-\lvert\chi_3+\chi_{13}\rvert^2-\chi_5+\chi_{11}\rvert^2$
$(\underline{5})$	$\lvert\chi_3+\chi_7+\chi_9+\chi_{13}\rvert^2+\lvert\chi_5+\chi_7+\chi_9+\chi_{11}\rvert^2-\lvert\chi_7+\chi_9\rvert^2+[(\chi_1+\chi_{15})\cdot(\overline{\chi_5}+\overline{\chi_{11}})+\text{h.c.}]$
$\underline{1}$	$(\chi_0+\chi_8+\chi_{16})\cdot(\overline{\chi_1}+\overline{\chi_{15}})+(\chi_4+\chi_8+\chi_{12})\cdot(\overline{\chi_3}+\overline{\chi_{13}})+(\chi_4+\chi_6+\chi_{10}+\chi_{12})\cdot(\overline{\chi_5}+\overline{\chi_{11}})+(\chi_2+\chi_6+\chi_8+\chi_{10}+\chi_{14})\cdot(\overline{\chi_7}+\overline{\chi_9})$
$(\underline{0})$	$\text{h.c.}(Z_{\underline{1}})$
$\underline{7}$	$(\chi_2+\chi_6+\chi_8+\chi_{10}+\chi_{14})\cdot(\sum_{i=0}^{7}(\overline{\chi_{2i+1}}))+(\chi_4+\chi_6+\chi_{10}+\chi_{12})\cdot(\sum_{i=1}^{6}(\overline{\chi_{2i+1}}))+(\chi_4+\chi_8+\chi_{12})\cdot(\overline{\chi_5}+\overline{\chi_7}+\overline{\chi_9}+\overline{\chi_{11}})+(\chi_0+\chi_8+\chi_{16})\cdot(\overline{\chi_7}+\overline{\chi_9})$
$(\underline{2})$	$\text{h.c.}(Z_{\underline{7}})$
$\underline{3}$	$(\chi_4+\chi_8+\chi_{12})\cdot(\sum_{i=0}^{7}(\overline{\chi_{2i+1}}))+(\chi_0+\chi_6+\chi_{10}+\chi_{16})\cdot(\overline{\chi_3}+\overline{\chi_{12}})+(\chi_2+\chi_6+\chi_{10}+\chi_{14})\cdot(\overline{\chi_5}+\overline{\chi_7}+\overline{\chi_9}+\overline{\chi_{11}})+(\chi_6+\chi_{10})\cdot(\overline{\chi_7}+\overline{\chi_9})+\chi_8\cdot(\overline{\chi_5}+\overline{\chi_{11}})$
$(\underline{4})$	$\text{h.c.}(Z_{\underline{3}})$
$\underline{5}$	$(\chi_4+\chi_6+\chi_{10}+\chi_{12})\cdot(\sum_{i=0}^{7}(\overline{\chi_{2i+1}}))+(\chi_2+2(\chi_8)+\chi_{14})\cdot(\sum_{i=1}^{6}(\overline{\chi_{2i+1}}))+(\chi_0+\chi_6+\chi_{10}+\chi_{16})\cdot(\overline{\chi_5}+\overline{\chi_{11}})+(\chi_4+\chi_6+\chi_{10}+\chi_{12})\cdot(\overline{\chi_7}+\overline{\chi_9})$
$(\underline{6})$	$\text{h.c.}(Z_{\underline{5}})$

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